DIMENSION OF METRIC SPACES AND HILBERT'S PROBLEM 13[1]

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In 1957 A. N. Kolmogorov [1] and V. I. Arnol'd [2] obtained the following result (answering Hilbert's conjecture in the negative):

THEOREM. *For every integer $n \geqq 2$ there exist continuous real functions ψ^{pq}, for $p=1, 2, \cdots, n$ and $q=1, 2, \cdots, 2n+1$, defined on the unit interval $E^1=[0, 1]$, such that every continuous real function f, defined on the n-dimensional unit cube E^n, is representable in the form*

$$f(x_1, \cdots, x_n) = \sum_{q=1}^{2n+1} \chi_q\left[\sum_{p=1}^{n} \psi^{pq}(x_p)\right],$$

where the functions χ_q are real and continuous.

The proof of the theorem relies on two properties of E^1, namely, E^1 is compact and of dimension 1. (By dimension we shall always mean covering dimension.) This paper generalizes the work of Kolmogorov and Arnol'd to obtain the following result:

THEOREM 2. *For $p=1, 2, \cdots, m$ let X^p be a compact metric space of finite dimension dp, and let $n=\sum_{p=1}^{n} d_p$. There exist continuous functions $\psi^{pq}: X^p \rightarrow [0, 1]$, for $p=1, \cdots, m$ and $q=1, 2, \cdots, 2n+1$, such that every continuous real function f defined on $\prod_{p=1}^{m} X^p$ is representable in the form*

$$f(x_1, \cdots, x_m) = \sum_{q=1}^{2n+1} \chi_q\left[\sum_{p=1}^{m} \psi^{pq}(x_p)\right],$$

where the functions χ_q are real and continuous.

The proof of Theorem 2 makes use of the following new characterization of dimension of metric spaces which is of interest in itself.

THEOREM 1. *A metric space X is of dimension $\leqq n$ if and only if for each open cover $\mathcal{C}$ of X and each integer $k \geqq n+1$ there exist k discrete families of open sets $\mathcal{U}_1, \cdots, \mathcal{U}_k$ such that the union of any $n+1$ of the $\mathcal{U}_i$ is a cover of X which refines $\mathcal{C}$.*

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By a discrete family of sets we mean a family such that each point has a neighborhood which meets at most one member of the family. For dimension 0 Theorem 1 reduces to the following result, which is well known: A metric space X is of dimension $\leqq 0$ if and only if each open cover of X may be refined by a discrete family of open sets.

This note presents brief proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. It suffices to show that if X is of dimension $\leqq n$, then for each open cover $\mathcal{C}$ of X and each integer $k \geqq n+1$ there exist k discrete families of open sets $\mathcal{U}_1, \cdots, \mathcal{U}_k$ which refine $\mathcal{C}$, any $n+1$ of which cover X. We prove this by induction on k. Let X be of dimension $\leqq n$ and let $\mathcal{C} = \{C_\alpha;\ \alpha \in I\}$ be an open cover of X. By passing to a refinement if necessary, we may suppose that $\mathcal{C}$ is locally finite. We may write X as $X = \bigcup_{i=1}^{n+1} X_i$ where each X_i is a subspace of dimension 0.

Let $\mathcal{C}_i = \{C_\alpha \cap X_i : \alpha \in I\}$. $\mathcal{C}_i$ is an X_i-open cover of X_i. By the case of dimension 0 there exists for each i a disjoint family $\mathcal{V}_i$ of X_i-open sets which covers X_i and refines $\mathcal{C}_i$.

Let I be well ordered by $<$. For each $\alpha \in I$ and $i = 1, \cdots, n+1$, let $W_\alpha^i = \bigcup\{V \in \mathcal{V}_i : V \subset C_\alpha \cap X_i$ and for each $\beta < \alpha$, $V \not\subset C_\beta \cap X_i\}$, and let $\mathcal{W}_i = \{W_\alpha^i : \alpha \in I\}$. Distinct members of $\mathcal{W}_i$ are disjoint.

Let $Z_\alpha^i = \{x \in C_\alpha : d(x, W_\alpha^i) < d(x, \bigcup_{\beta<\alpha} W_\beta^i)\}$ and let $\mathcal{Z}_i = \{Z_\alpha^i : \alpha \in I\}$. $W_\alpha^i \subset Z_\alpha^i \subset C_\alpha$ for each i and α. Each $\mathcal{Z}_i$ is a locally finite family of disjoint open sets of X which refines $\mathcal{C}$ and covers X_i. It follows that $\bigcup_{i=1}^{n+1} \mathcal{Z}_i$ covers X.

As before, there exist closed sets $D_\alpha^i \subset Z_\alpha^i$ such that $\{D_\alpha^i : \alpha \in I;\ i = 1, \cdots, n+1\}$ covers X. Choose open sets U_α^i such that $D_\alpha^i \subset U_\alpha^i \subset \mathrm{Cl}(U_\alpha^i) \subset Z_\alpha^i$, and let $\mathcal{U}_i = \{U_\alpha^i : \alpha \in I\}$. Each $\mathcal{U}_i$ is discrete and refines $\mathcal{C}$, and $\mathcal{U}_1, \cdots, \mathcal{U}_{n+1}$ cover X.

Suppose now that $k > n+1$ and that $\mathcal{U}_1, \cdots, \mathcal{U}_{k-1}$ are discrete families of open sets which refine $\mathcal{C}$, any $n+1$ of which cover X. We will construct a subset A of X and a discrete family $\mathcal{U}_k$ of open sets which refines $\mathcal{C}$, such that any n of the families $\mathcal{U}_1, \cdots, \mathcal{U}_{k-1}$ cover $X - A$ and $\mathcal{U}_k$ covers A.

Let $\mathfrak{A} = \{\gamma = (\gamma_1, \cdots, \gamma_n) : 1 \leqq \gamma_1 < \gamma_2 < \cdots < \gamma_n \leqq k-1\}$. For $\gamma \in \mathfrak{A}$, let $A_\gamma = \bigcap_{i=1}^n (X - \bigcup \mathcal{U}_{\gamma_i})$, and $A = \bigcup_{\gamma \in \mathfrak{A}} A_\gamma$. Each A_γ is closed and $A_\gamma \cap A_\delta = \emptyset$ for $\gamma \neq \delta$. Hence there exist open sets B_γ such that $A_\gamma \subset B_\gamma$ and $\mathrm{Cl}(B_\gamma) \cap \mathrm{Cl}(B_\delta) = \emptyset$ for $\gamma \neq \delta$. For $\gamma \in \mathfrak{A}$, there exists a positive integer $j_\gamma \leqq k-1$ such that $j_\gamma \notin \{\gamma_i : i = 1, 2, \cdots, n\}$. $\mathcal{U}_{j_\gamma}$ covers A_γ. Let $\mathcal{U}_k = \{U \cap B_\gamma : \gamma \in \mathfrak{A}$ and $U \in \mathcal{U}_{j_\gamma}\}$. $\mathcal{U}_k$ is a discrete family of open sets which covers A and refines $\mathcal{C}$. Thus the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. For each integer p, $p = 1, \cdots, m$, each integer q, $q=1, 2, \cdots, 2n+1$, and each integer k, $k=1, 2, \cdots$, there exist positive real numbers γ_k and ϵ_k, distinct positive prime numbers r_k^{pq}, discrete families $\mathcal{S}_k^{pq}$ of open sets of X^p, and continuous functions $f_k^{pq}: X^p \to [0, 1]$ such that:

(1) $\lim_{k\to\infty} \gamma_k = \lim_{k\to\infty} \epsilon_k = 0$;

(2) each member of $\mathcal{S}_k^{pq}$ is of diameter $\leqq \gamma_k$ and for each fixed p and k any d_p+1 of the families $\mathcal{S}_k^{pq}$ cover X^p;

(3) $m\epsilon_k < 1/\prod_{p=1}^m r_k^{pq}$ for each $q=1, 2, \cdots, 2n+1$;

(4) f_k^{pq} is constant on each member of $\mathcal{S}_k^{pq}$, the constant being an integral multiple of $1/r_k^{pq}$, and takes different values on distinct members of $\mathcal{S}_k^{pq}$;

(5) For each $j<k$ and $x \in X^p$, $f_j^{pq}(x) \leqq f_k^{pq}(x) \leqq f_j^{pq}(x) + \epsilon_j - \epsilon_k$.

The γ_k, ϵ_k, r_k^{pq}, $\mathcal{S}_k^{pq}$, and f_k^{pq} are defined inductively on k.

Let $X = \prod_{p=1}^m X^p$. X is a metric space with metric

$$d((x_1, \cdots, x_m), (y_1, \cdots, y_m)) = \sum_{p=1}^{m} d(x_p, y_p).$$

For each q and k let $\mathfrak{I}_k^q = \{\prod_{p=1}^m C^p : C^p \in \mathcal{S}_k^{pq} \text{ for each } p\}$. Each $\mathfrak{I}_k^q$ is a discrete family of open sets of X, and each member of $\mathfrak{I}_k^q$ is of diameter $\leqq m\gamma_k$. For each k any $n+1$ of the families $\mathfrak{I}_k^q$ cover X.

Let $\psi^{pq}(x) = \lim_{k\to\infty} f_k^{pq}(x)$ for $x \in X^p$. For each k and each $x \in X^p$, $f_k^{pq}(x) \leqq \psi^{pq}(x) \leqq f_k^{pq}(x) + \epsilon_k$. Thus ψ^{pq}, being the uniform limit of the f_k^{pq}, is continuous.

Let $\phi^q(x_1, \cdots, x_m) = \sum_{p=1}^m \psi^{pq}(x_p)$ for $(x_1, \cdots, x_m) \in X$. Let $\mathcal{U}_k^q = \{\phi^q(C): C \in \mathfrak{I}_k^q\}$. If $C = \prod_{p=1}^m C^p \in \mathfrak{I}_k^q$, then $\phi^q(C)$ is contained in the interval $[\sum_{p=1}^m f_k^{pq}(C^p), \sum_{p=1}^m f_k^{pq}(C^p) + m\epsilon_k]$. By condition (3) these closed intervals are disjoint for each fixed q and k. Hence each $\mathcal{U}_k^q$ is discrete.

Let f be a continuous real-valued function on X. For each integer $r \geqq 0$ and $q=1, \cdots, 2n+1$ there exists a positive integer k_r and continuous functions $\chi_r^q: R \to R$ (R denotes the real line, $k_0=1$ and $\chi_0^q=0$ for each q) such that if $f_r(x) = \sum_{q=1}^{2n+1} \sum_{s=0}^{r} \chi_s^q(\phi^q(x))$ for $x \in X$ and if $M_r = \sup_{x\in X} |(f-f_r)(x)|$, then:

(6) $k_{r+1} > k_r$;

(7) if $d(a, b) < m\gamma_{k_{r+1}}$, then $|(f-f_r)(a) - (f-f_r)(b)| < (2n+2)^{-1}M_r$;

(8) χ_{r+1}^q is constant on each member of $\mathcal{U}_{k_{r+1}}^q$, its value on $\phi^q(C)$ for $C \in \mathfrak{I}_{k_{r+1}}^q$ being $(n+1)^{-1}(f-f_r)(y)$ for some arbitrarily chosen element y of C;

(9) $|\chi_{r+1}^q(a)| \leqq (n+1)^{-1}M_r$ for each $a \in R$.

The k_r and χ_r^q are defined inductively on r. It is easily deduced from (7) and (8) that

(10) $|(n+1)^{-1}(f-f_r)(x)-\chi_{r+1}^q(\phi^q(x))|<(n+1)^{-1}(2n+2)^{-1}M_r$ for $x\in\bigcup\{T: T\in\mathfrak{J}_{k,r+1}^q\}$.

For each $x\in X$ there are at least $n+1$ distinct values of q such that that $x\in\bigcup\{T: T\in\mathfrak{J}_{k,r+1}^q\}$. Adding (10) for $n+1$ values of q and (9) for the other n values of q yields

$$|(f-f_{r+1})(x)| = \left|(f-f_r)(x) - \sum_{q=1}^{2n+1}\chi_{r+1}^q(\phi^q(x))\right| < \frac{2n+1}{2n+2}M_r.$$

Then $M_{r+1}<(2n+1)(2n+2)^{-1}M_r$, so $M_r<((2n+1)(2n+2)^{-1})^r M_0$ for each r and $\lim_{r\to\infty} M_r=0$. Hence $f(x)=\lim_{r\to\infty} f_r(x)$ for all $x\in X$. Moreover, by condition (9) the functions $\sum_{s=0}^n \chi_s^q$ converge uniformly for each q to a continuous function $\chi^q: R\to R$ and

$$f(x) = \lim_{r\to\infty} f_r(x) = \lim_{r\to\infty}\sum_{q=1}^{2n+1}\sum_{s=0}^{r}\chi_s^q(\phi^q(x)) = \sum_{q=1}^{2n+1}\chi^q(\phi^q(x)).$$

This completes the proof of Theorem 2.

References

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