A GENERALIZATION OF THE HILTON-MILNOR THEOREM

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The Hilton-Milnor theorem states that $\Omega\bigvee_{i=1}^{n} \Sigma X_i$ is homotopy equivalent to a weak infinite product, $\prod_{i=1}^{\infty} \Omega\Sigma X_i$, where each X_i, $i>n$, is a smash product of the X_i's, $i \leqq n$. In this note we extend this theorem to the 'wedges' lying between $\bigvee_{i=1}^{n} \Sigma X_i$ and $\prod_{i=1}^{n} \Sigma X_i$.

It will be assumed that all spaces are connected countable CW-complexes with base points. $T_i(X_1, \cdots, X_n)$ is the subset of $X_1 \times \cdots \times X_n$ consisting of those points with at least i coordinates at base points. T_0 is the cartesian product and T_{n-1} is the space studied by Hilton and Milnor. T_{n-1} will also be denoted by $\bigvee_{j=1}^{n} X_j$. The smash product $\wedge(X_1, \cdots, X_n)$ is the quotient space $T_0(X_1, \cdots, X_n)/T_1(X_1, \cdots, X_n)$. Define $X^{(n)}$ inductively by $X^{(0)} = S^0$ and $X^{(n)} = \wedge(X^{(n-1)}, X)$, for $n>0$.

The n-fold suspension, $\Sigma^n X$, is defined to be $\wedge(S^n, X)$. The loop space of X, ΩX, is the set of maps, $f\colon I \to X$, such that $f(0) = f(1) = *$. We shall abbreviate $(\Sigma X_1, \cdots, \Sigma X_n)$ and $(\Omega X_1, \cdots, \Omega X_n)$ by $\Sigma(X_1, \cdots, X_n)$ and $\Omega(X_1, \cdots, X_n)$, respectively.

THEOREM 1. *$\Omega T_i \Sigma(X_1, \cdots, X_n)$ is homotopy equivalent to a weak infinite product, $\prod_{j=1}^{\infty} \Omega\Sigma X_j$, where each X_j is equal to $\Sigma^r \wedge (X_1^{(j_1)}, \cdots, X_n^{(j_n)})$ for some $(n+1)$-tuple, $(r, j_1, \cdots, j_n)$, depending upon j. Moreover, the set of $(n+1)$-tuples over which the product is taken is computable.*

If $i = n-1$, Theorem 1 is the Hilton-Milnor theorem. It was proven in [1] by Hilton when the X_i are spheres and extended to the general case by Milnor [2].

We shall sketch the proof of Theorem 1, when $n - i \geqq 2$. The details will appear in [3].

The inclusion map $j\colon T_i(X_1, \cdots, X_n) \to T_0(X_1, \cdots, X_n)$ may be replaced by a homotopy equivalent fibre map, $p\colon E \to T_0$, with fibre F_i. It is easily seen that when $n - i \geqq 2$, the short exact sequence

$$* \to \Omega F_i \to \Omega E \to \Omega T_0 \to *$$

splits yielding:

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LEMMA 1. $\Omega T_i(X_1, \cdots, X_n) \sim \Omega X_1 \times \cdots \times \Omega X_n \times \Omega F_i$.

Thus an analysis of ΩT_i depends upon a study of F_i. Standard homotopy methods are applied and it is shown that

THEOREM 2. *F_i is homotopy equivalent to*

$$\bigvee_S \bigvee^r (\Sigma^{n-i} \wedge \Omega(X_{j_1}, \cdots, X_{j_k}))$$

with $S = \{(j_1, \cdots, j_k) \mid 1 \leqq j_1 < \cdots < j_k \leqq n$ with $n-i+1 \leqq k \leqq n\}$ and r equal to the binomial coefficient

$$\binom{k-1}{n-i}$$

where $\vee^r X$ is the one point union of r copies of X.

If we rename the spaces of Theorem 2, we may write $\Omega F_i \sim \Omega \bigvee_{j=1}^{N} \Sigma Y_j$. This is the case studied by Hilton and Milnor. Their result shows that ΩF_i is homotopy equivalent to a weak infinite product, $\prod_{j=1}^{\infty} \Omega\Sigma Y_j$, where each $Y_j = \Sigma^r \wedge (Y_1^{(i_1)}, \cdots, Y_N^{(i_N)})$ for some $(N+1)$-tuple, $(r, i_1, \cdots, i_N)$. Since each Y_j, $j \leqq N$, is of the form $\Sigma^{n-i-1} \wedge ((\Omega X_1)^{(i_1)}, \cdots, (\Omega X_n)^{(i_n)})$, it follows that each Y_j, $j > N$, is of the form $\Sigma^r \wedge ((\Omega X_1)^{(j_1)}, \cdots, (\Omega X_n)^{(j_n)})$. We thus have:

THEOREM 3. *$\Omega T_i(X_1, \cdots, X_n)$ is homotopy equivalent to a weak infinite product, $\prod_{j=1}^{\infty} \Omega\Sigma X_j$, where each X_j, $j > n$, equals*

$$\Sigma^r \wedge ((\Omega X_1)^{(j_1)}, \cdots, (\Omega X_n)^{(j_n)})$$

for some $(n+1)$-tuple, $(r, j_1, \cdots, j_n)$, depending upon j. In addition there exists an algorithm for computing the set of $(n+1)$-tuples over which the product is taken.

In particular the algorithm is given by combining the Hilton-Milnor theorem with Theorem 2. Note that the X_i, $i \leqq n$, of Theorem 3 need not be suspensions. However, if each $X_i = \Sigma Y_i$, for some space Y_i, a further decomposition is possible as seen by the following theorem.

THEOREM 4. *If $r \geqq 1$, $\Omega\Sigma^r \wedge \Omega\Sigma(Y_1, \cdots, Y_m)$ is homotopy equivalent to a weak infinite product, $\prod_{i=m+1}^{\infty} \Omega\Sigma Y_i$, where each Y_i, $i \geqq m+1$, is equal to $\Sigma^t \wedge (Y_1^{(i_1)}, \cdots, Y_m^{(i_m)})$ for some $(m+1)$-tuple, $(t, i_1, \cdots, i_m)$. Moreover, an explicit algorithm can be given for computing the set of $(m+1)$-tuples over which the product is taken.*

Theorem 1 follows from Theorems 3 and 4.

The proof of Theorem 4 is modeled after [2]. The set of Y_j, $j>m$, of Theorem 4 is called a set of $\wedge$*-basic products* and is defined inductively as follows. The basic product of weight one are $Y_1, \cdots, Y_m$, and $\Sigma^{m-1}\wedge(Y_1, \cdots, Y_m)=Y_{m+1}$. Those of weight two are $Y_{m+j+1}=\wedge(Y_{m+1}, Y_j)$, $j=1, \cdots, m$. Define e by setting $e(h)=0$ if $1\leqq h\leqq m+1$ and $e(h)=h-(m+1)$ if $m+1<h\leqq 2m+1$. Let $n>2$. Assume inductively that the products of weight less than n have been defined and are ordered and that $e(i)$ is defined for all such i. The basic products of weight n are all elements $\wedge(Y_i, Y_j)$ such that weight Y_i+weight $Y_j=n$ and $e(i)\leqq j<i$. These are ordered arbitrarily among themselves and are greater than all products of lesser weight. Let $e(h)=j$ if $Y_h=\wedge(Y_i, Y_j)$. This completes the inductive description of $\{Y_j\}$.

REFERENCES

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