THE OBSTRUCTION TO THE LOCALIZABILITY OF A MEASURE SPACE[1]

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This paper, an outgrowth of the author's doctoral dissertation,[2] presents a necessary and sufficient condition, of a cohomological nature, for a measure space to be localizable in the sense of Segal.[3] In order to state the main theorem, we must fix some terminology and establish some notation.

1. **Definitions.**[4] A *measure space* (X, R, m) consists of a set X, a boolean ring R of subsets of X, and a finite, nonnegative, finitely additive measure m on R subject to the requirement:

$$\{E_n \in R \ (n = 1, 2, \cdots), E_n \cap E_k = \varnothing \ (n \neq k), \\ \Sigma_n m(E_n) < \infty, \ E = \bigcup_n E_n\} \Rightarrow \{E \in R \text{ and } m(E) = \sum_n m(E_n)\}.$$

If (X, R, m) is a measure space, a subset K of X is *measurable* if $K \cap E \in R$ whenever $E \in R$; it is *null* if it is measurable and $m(K \cap E) = 0$ whenever $E \in R$. The *measure ring* $\mathfrak{M}$ of the measure space (X, R, m) is the quotient of the (sigma ring of) measurable sets by the (sigma ideal of) null sets. A measure space is *localizable* if its measure ring is complete as a partially ordered set.

2. Let (X, R, m) be a measure space. Consistent use will be made of the following notation:

I: the ideal of sets $K \in R$ for which $m(K) = 0$;
M_1: the sigma ring of measurable sets;
X_R: the set $\bigcup R \in M_1$;
M: the principal ideal of M_1 determined by X_R;
N_1: the sigma ideal of null sets in M_1;
N: the sigma ideal of null sets in M, i.e., $M \cap N_1$;

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[2] F. E. J. Linton, *The functorial foundations of measure theory*, Columbia Univ., New York, 1963.

[3] I. E. Segal, *Equivalences of measure spaces*, Amer. J. Math. **73** (1951), 275–313.

[4] This is merely a restatement, for the convenience of the reader, of parts of Definitions 2.1, 2.2, 2.4, and 2.6 in the cited work of Segal. Incidentally, our rings need not have unit elements.

r: the countably additive measure on M_1 defined by $r(K) = \sup_{E \in R} m(K \cap E)$.

3. The following observations are easily verified.[5]

(3.1) For $K \in M_1$, $r(K) = 0$ iff $K \in N_1$.

(3.2) Every subset of X disjoint from X_R is null.

(3.3) r and m take the same values on R.

(3.4) $\mathfrak{M} \cong M_1/N_1 \cong M/N$.

4. **The obstruction.** Let J be an ideal in a boolean ring A. Since the intersection of an element of J with an element of A is again an element of J, we may consider J as a sub-A-module of the A-module A; in consequence, we may also think of the quotient ring A/J as an A-module—this A-module structure on A/J is the same as that induced from the natural (A/J)-module structure by change of rings using the canonical projection $A \to A/J$. Let $d_{A,J}$ denote the connecting homomorphism[6]

$$d_{A,J}\colon \mathrm{Hom}_A(A, A/J) \to \mathrm{Ext}_A^1(A, J)$$

between the group of A-module homomorphisms $A \to A/J$ and the group of A-module extensions $0 \to J \to ? \to A \to 0$. We call $d_{A,J}$ the *obstruction to the localizability of the boolean ring A over its ideal J.* If $d_{A,J} = 0$, we say *A is localizable over J.*

5. **Main theorem.** *The measure space (X, R, m) is localizable if and only if the obstruction $d_{R,I}$ to the localizability of R over I vanishes.*

It is to be noted that no question of higher obstructions arises.

6. **The ring βA.** The proof of Theorem 5 depends on information regarding the largest boolean ring βA containing a given boolean ring A as a dense ideal.[7] The uniqueness of βA is due to the fact that its Stone space must be the Stone-Čech compactification of the Stone space of A. Its existence is demonstrated by proving that the clopen sets in that compactification have the desired property. Alternate equivalent descriptions of βA are:

(6.1) βA is the ring of clopen sets in the Stone space of A (A appears as the ring of *compact* open sets);

(6.2) if A is represented as a ring of subsets of a set Z, then $\beta A \cong \{ Y \mid Y \subseteq \bigcup A,\ Y \cap a \in A \text{ for all } a \in A \}$;

[5] Consult Segal, *op. cit.*, for proofs of (3.1) and (3.3); the proof of (3.2) is immediate, and (3.4) follows from the rest.

[6] Cf. S. Mac Lane, *Homology*, p. 74, Springer, Berlin, 1963.

[7] Some of the material presented in §6 is contained in §1.10 of the author's dissertation.

(6.3) $\beta A \cong \mathrm{Hom}_A(A, A)$ (the ring of A-module endomorphisms of A);

(6.4) βA is the inverse limit of the inverse system

$$(\{A_a\}_{a\in A},\quad \{p_{a,b}: A_a \to A_b\}_{a\geq b})$$

of all principal ideals $A_a = \{x \mid x \leqq a\}$ of A, where $p_{a,b}(x) = x \wedge b$.

7. **The main lemma.** If J is an ideal in the boolean ring A, the canonical projection $A \to A/J$ induces three maps

$$(7.1)\qquad \begin{array}{ccc} \mathrm{Hom}_A(A, A/J) & \leftarrow & \mathrm{Hom}_A(A/J, A/J) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_A(A, A) & \dashrightarrow & \mathrm{Hom}_{A/J}(A/J, A/J) \end{array}$$

by covariant composition, contravariant composition, and change of rings, respectively. The second and third maps are isomorphisms. The indicated inverse composite is easily seen to be a unitary ring homomorphism; this, when combined with the representations (6.3) of βA and $\beta(A/J)$, yields a distinguished map

$$(7.2)\qquad \beta A \to \beta(A/J),$$

about which the essential information is recorded in the lemma below. The first statement of this lemma is clear from the discussion above; the remaining statements depend only upon the exactness[8] of the sequence

$$0 \to \mathrm{Hom}_A(A, J) \to \mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(A, A/J) \xrightarrow{d_{A,J}} \mathrm{Ext}^1_A(A, J).$$

(7.3) MAIN LEMMA. *The map* (7.2) *is a boolean homomorphism. In the representation* (6.3), *its kernel is given by* $\mathrm{Hom}_A(A, J)$; *using* (6.2), *instead, its kernel is the family*

$$\tilde{J} = \{Y \mid Y \subseteq \cup A,\ Y \cap a \in J \text{ for all } a \in A\}.$$

Moreover, the induced monomorphism $\beta A/\tilde{J} \to \beta(A/J)$ *is an isomorphism if and only if* A *is localizable over* J.

8. Although (7.3) is the main tool used in the proof of Theorem 5, a few simple observations must be made before it can successfully be applied. Namely, let (X, R, m) be a measure space. Then:

(8.1) $M \cong \beta R$ (consequence of (6.2));

(8.2) $N \cong \tilde{I}$ (consequence of (7.3));

(8.3) $\mathfrak{M} \cong \beta R/\tilde{I}$ (consequence of (3.4), (8.1), (8.2));

[8] Cf. Mac Lane, *loc. cit.*

(8.4) $R/I \subseteq \mathfrak{M} \subseteq \beta(R/I)$ (consequence of (3.3), (7.3), (8.3));

(8.5) $\beta(R/I)$ is complete (consequence of (6.4), the completeness of each principal ideal in R/I, and the fact that each map in the inverse system (6.4) is a complete homomorphism).

9. **Proof of Theorem 5.** According to Lemma (7.3), when we take into account (8.3) and (8.4), a necessary and sufficient condition for R to be localizable over I is that the inclusion $\mathfrak{M} \subseteq \beta(R/I)$ be the identity. If, indeed, this is the identity, (8.5) assures that $\mathfrak{M}$ is complete, so that (X, R, m) is localizable. If, conversely, (X, R, m) is localizable, $\mathfrak{M}$ is complete, and since complete boolean rings are injective,[9] the inclusion $\mathfrak{M} \subseteq \beta(R/I)$ admits a retraction $p\colon \beta(R/I) \to \mathfrak{M}$. In order to prove $\mathfrak{M} = \beta(R/I)$, it suffices to know that this retraction is one-one. So let $b \in \beta(R/I)$, and assume $b \neq 0$. Then b contains a nonzero element $a \in R/I$, and, by (8.4), $p(a) \neq 0$ (indeed, $p(a) = a$). But $p(b) \geqq p(a)$, since $b \geqq a$, and so $p(b) \neq 0$. Thus p is one-one, $\mathfrak{M} = \beta(R/I)$, and the proof is complete.

10. **Localizability and the dual of L_1.** Theorem 5 can be used to deliver a quick and revealing proof of Segal's theorem[10] that the measure space (X, R, m) is localizable if and only if the usual "integral of the product" map from the Banach space L_∞ of (essentially) bounded M_1-measurable functions mod N_1-measurable functions to the dual of the space $L_1 = L_1(X, R, m)$ is an isomorphism. For by an extension of a theorem[11] of Sikorski, L_∞ is isomorphic with the space[12] of bounded Carathéodory functions on $M_1/N_1 \cong \mathfrak{M} \cong \beta R/\tilde{I}$. On the other hand, the dual of the space L_1 is[13] the space of bounded Carathéodory functions on $\beta(R/I)$, which, because $\beta R/\tilde{I} \subseteq \beta(R/I)$, contains L_∞ as an isometrically embedded subspace. Consequently, L_∞ is the dual of L_1 if and only if these two spaces of Carathéodory functions coincide, and this, in turn, is the case if and only if $\beta R/\tilde{I}$ and $\beta(R/I)$ coincide, i.e., using Theorem 5 and Lemma (7.3), if and only if (X, R, m) is localizable.

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[9] Cf. P. R. Halmos, *Injective and projective boolean algebras*, Proc. Sympos. Pure Math., Vol. 2, pp. 114–122, Amer. Math. Soc. Providence, R. I., 1961.

[10] Segal, *op. cit.*

[11] R. Sikorski, *Boolean algebras*, Proposition 32.5, Springer, Berlin, 1960.

[12] Cf. C. Goffman, *Remarks on lattice-ordered groups and vector lattices*. I. *Carathéodory functions*, Trans. Amer. Math. Soc. 88 (1958), 107–120.

[13] Theorem (2.5.10) of the author's dissertation.