THE GAUSS-BONNET THEOREM AND THE TAMAGAWA NUMBER

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This note is an outline of some of the author's recent work on the Tamagawa numbers. Our purpose is to express the Tamagawa number of Chevalley groups in terms of Euler-Poincaré characteristics, Bernoulli numbers and Gaussian curvatures.

The method can, of course, be applied to derive other formulas of the same type. We treat, however, only Chevalley groups, because it is in the present case that the main ideas of our method are most clear. The details and further results will be given elsewhere.

Let G_0 be a connected semisimple real Lie group without compact factors and center, $\mathfrak{g}_0$ its Lie algebra and put $\mathfrak{g} = \mathfrak{g}_0 \otimes C$. Let K be a maximal compact subgroup of G_0 and let $\mathfrak{k}$ be its Lie algebra. To the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p}$ being the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form B of $\mathfrak{g}$, there corresponds the dual $\mathfrak{u} = \mathfrak{k} + (-1)^{1/2}\mathfrak{p}$. Denote by U the connected compact group whose Lie algebra is $\mathfrak{u}$ and by T a maximal torus of U. We denote by $dG_0, dU,$ $dK, dT, d(G_0/K), d(U/K), d(U/T)$ the volume elements corresponding to the Riemannian structure on each space obtained naturally by B (cf. [4, Chapter IV]). We also denote by $\kappa(M)$ the Gaussian curvature of a Riemannian manifold M and by $\chi(M)$ the Euler-Poincaré characteristic. One has $\kappa(U/K) = (-1)^{m/2}\kappa(G_0/K)$ with $m = \dim(G_0/K)$. By the Gauss-Bonnet theorem (cf. [1], [2]), we have

$$(1) \qquad \frac{2\kappa(U/K)}{A_m} \int_{U/K} d(U/K) = \chi(U/K),$$

$$(2) \qquad \frac{2\kappa(U/T)}{A_{2N}} \int_{U/T} d(U/T) = \chi(U/T) = [W],$$

where A_m is the surface area of the unit m-sphere, N is the number of positive roots of $\mathfrak{g}$ and $[W]$ is the order of the Weyl group W of $\mathfrak{g}$. Eliminating the volume of U from (1), (2), one gets

$$(3) \qquad \int_K dK = \int_T dT \, \frac{A_{2N}[W]\kappa(U/K)}{A_m \chi(U/K)\kappa(U/T)} \,.$$

Let Γ be a discrete subgroup of G_0 such that

$$\int_{\Gamma \backslash G_0} dG_0 < + \infty.$$

We define the "Euler-Poincaré characteristic" of $\Gamma \backslash G_0 / K$ by

(4) $\dfrac{2\kappa(G_0/K)}{A_m} \displaystyle\int_{\Gamma \backslash G_0 / K} d(G_0/K) = \chi(\Gamma \backslash G_0 / K).$

From (3), (4), we have

(5) $\displaystyle\int_{\Gamma \backslash G_0} dG_0 = (-1)^{m/2} \int_T dT \, \dfrac{A_{2N}[W] \chi(\Gamma \backslash G_0 / K)}{2\chi(U/K)\kappa(U/T)}.$

Now, let $G \subset GL(n)$ be a connected semisimple algebraic group defined over Q. Let $\mathfrak{g}$ be the Lie algebra of G and let X_i, $1 \leq i \leq n$, be a basis of $\mathfrak{g}$ rational over Q. Call ω the algebraic n-form over Q which is the product of 1-forms dual to X_i. Let G_0 be the identity component of G_R, and let $\mathfrak{g}_0$ be the Lie algebra of G_0. Then ω induces on G_0 a volume element ω_R, and we see that

(6) $dG_0 = (\det B^+(X_i, X_j))^{1/2} \omega_R,$

where $B^+(X, Y) = -B(X, \sigma Y)$, σ being the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$.

Assume now that G is the Chevalley group, i.e., the identity component of the group of automorphisms of a complex semisimple Lie algebra $\mathfrak{g}$. With respect to a Chevalley basis of $\mathfrak{g}$ (cf. [3, Théorème 1]) G becomes an algebraic subgroup of $GL(n)$ defined over Q, $n = \dim \mathfrak{g}$. By choosing suitable K and T, and putting $\Gamma = G_Z \cap G_0$, here G_Z being the group of units of G, we get from (5), (6) the following:

(7)

$$\int_{G_Z \backslash G_R} \omega_R = \int_{\Gamma \backslash G_0} \omega_R$$

$$= \dfrac{(-1)^{(l+N)/2} 2^{l+N} N![W]}{(2N)!C} \prod_{r>0} \langle r, r \rangle \, \dfrac{\chi(\Gamma \backslash G_0 / K)}{\chi(U/K)\kappa(U/T)} \pi^{l+N},$$

where $\langle r, r \rangle$ is the inner product defined by B, C is the determinant of the Cartan matrix and l is the rank of $\mathfrak{g}$.

On the other hand, ω induces a measure ω_p on G_{Q_p} for all prime numbers p. Denote by G_{Z_p} the group of p-adic units of G_{Q_p}. By the results of Iwahori-Matsumoto (cf. [5, Corollary 2.17]) and Ree (unpublished) on p-adic Chevalley groups, we get

$$\int_{G_{Z_p}} \omega_p = \frac{1}{|C|_p} \frac{[G_{F_p}]}{p^n} \quad \text{for all } p,$$

where G_{F_p} is the finite group of rational points over F_p. Since $\prod_v |C|_v = 1$, we get

$$(8) \qquad \prod_p \int_{G_{Z_p}} \omega_p = C \prod_{i=1}^{l} \zeta(a_i)^{-1},$$

where the a_i are integers such that the Poincaré polynomial of the compact form of G is $\prod_{i=1}^{l} (t^{2a_i-1}+1)$ (cf. [3, p. 46]). When a_i are all even, we have

$$\zeta(a_i) = 2^{a_i-1} \frac{\pi^{a_i}}{a_i!} B_{a_i/2} \qquad (1 \leq i \leq l).$$

Since $\sum_{i=1}^{l} a_i = l+N$ and the class number of our G is 1, we get, from (7) and (8), the following:

$$(9) \quad \tau(G) = \frac{(-1)^{(l+N)/2} 2^l N![W]}{(2N)!} \prod_{i=1}^{l} a_i! B_{a_i/2}^{-1} \prod_{r>0} \langle r, r \rangle \frac{\chi(\Gamma \backslash G_0/K)}{\chi(U/K)\kappa(U/T)},$$

with $\Gamma = G_Z \cap G_0$.

REMARK 1. Our method does not apply if m is odd or $\chi(U/K)=0$ or a_i is odd for some i. When $\mathfrak{g}$ is simple, these happen for groups of type A_l ($l \geq 2$), D_l (l odd) and E_6 (cf. [3, p. 64]). The first two cases are classical and the $\tau(G)$ are known (cf. [9]). Therefore E_6 is the only exception among Chevalley groups. It is interesting to notice that $\chi(U/K) \neq 0$ if and only if a_i are all even. This would explain the difficulty of finding $\zeta(2n+1)$.

REMARK 2. On the right-hand side of (9), all factors except $\chi(\Gamma \backslash G_0/K)$ are $\in Q$. As for $\chi(\Gamma \backslash G_0/K)$, one could probably prove (even for the general case) that $\chi(\Gamma' \backslash G_0/K)$ is $\in Z$, Γ' being a normal subgroup of Γ with finite index, by making use of Allendoerfer-Weil's theorem as Satake has done for symplectic groups (cf. [1], [6]).

REMARK 3. For general semisimple groups, the difficulty arises from the p-adic part rather than from the ∞-adic part. Namely, the zeta function should be replaced by L-functions and one has to know their values at some integers.

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