COBORDISM CLASSES OF SQUARES OF ORIENTABLE MANIFOLDS

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In this paper we give an outline of the following theorem.[1] Full details will appear elsewhere.

THEOREM. *If M is an orientable manifold, then there exists a spin manifold N such that N is cobordant to $M \times M$ (in the unoriented sense). (For definitions and notation see* [1] *and* [3].)

Following C. T. C. Wall [5] we construct a set of orientable manifolds whose cobordism classes generate the image of the "orientation ignoring homomorphism" $r: \Omega \to \mathfrak{N}$, and the theorem is then verified for each of these generators.

Some of these manifolds are certain complex projective spaces CP^n. As was noted in [2], $CP^n \times CP^n$ is cobordant to quaternionic projective space HP^n. Since HP^n is always 3-connected it is a spin manifold.

A second type of manifold used is constructed as follows. Let λ be the canonical nontrivial line bundle over real projective space P^n, and ϵ^m the trivial m-plane bundle over P^n. Define $M(m, n)$ as the space of lines through the origin in each fibre of the Whitney-sum bundle $\lambda \oplus \epsilon^n$. $M(m, n)$ is an orientable manifold if and only if m is odd and n is even, and certain of these manifolds are used as generators for $r(\Omega)$.

The third type of manifold used is denoted by

$$M(m_1, n_1; m_2, n_2; \cdots ; m_{r+1}, n_{r+1}),$$

where $r \geqq 1$, m_i is odd and n_i is even for $i = 1, \cdots, r+1$. This manifold is the total space of a certain fibre bundle over $S^1 \times \cdots \times S^1$ (r factors), with fibre $M(m_1, n_1) \times \cdots \times M(m_{r+1}, n_{r+1})$.

To prove the theorem for these last two types of manifolds we construct their "complex analogues" as follows. Let $c\lambda$ denote the canonical complex line-bundle over complex projective space CP^n, and $c\epsilon^m$ the trivial complex m-plane bundle over CP^n. Then $CM(m, n)$ is the space of complex lines through the origin in each fibre of $c\lambda \oplus c\epsilon^m$. $CM(m_1, n_1; \cdots ; m_{r+1}, n_{r+1})$ will be the total space of a fibre

[1] This theorem was originally conjectured by J. Milnor in [2].

bundle over $S^2 \times \cdots \times S^2$ (r factors) with fibre $CM(m_1, n_1) \times \cdots \times CM(m_{r+1}, n_{r+1})$.

If M is one of our manifolds we use the following method to verify that $M \times M$ is cobordant to CM. $H^*(M; Z_2)$ and $H^*(CM; Z_2)$ are isomorphic truncated polynomial algebras over Z_2, the former on several one-dimensional generators and the latter on several two-dimensional generators. If we represent this isomorphism by D: $H^*(M) \to H^*(CM)$ (note that $D(H^i(M)) = H^{2i}(CM)$), we may prove that D preserves Stiefel-Whitney classes: $D(w(M)) = w(CM)$. Since $w_1(M) = 0$, $w_1(CM) = w_2(CM) = 0$; hence CM is a spin manifold.

Let $w_{i_1} \cdots w_{i_k}[M]$ denote a typical Stiefel-Whitney number of M; then by a theorem of Wall [5]

$$w_{i_1} \cdots w_{i_k}[M] = w_{2i_1} \cdots w_{2i_k}[M \times M],$$

and $w_{j_1} \cdots w_{j_h}[M \times M] = 0$ if any of the j's is odd. Therefore

$$w_{i_1} \cdots w_{i_k}[M \times M] = w_{i_1} \cdots w_{i_k}[CM]$$

for any $i_1, \cdots, i_k$. So by R. Thom's result [4], $M \times M$ is cobordant to CM.

Bibliography

1. J. Milnor, *Lectures on characteristic classes*, Princeton University, Princeton, N. J., 1957.

2. ———, *On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology (to appear).

3. ———, *Spin structures on manifolds*, Enseignement Math. **9** (1963), 198–203.

4. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

5. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

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