# A SIMPLE PROOF OF THE THEOREM OF P. J. COHEN

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Cohen, in [1], completing a line of investigations initiated by Helson [2] and Rudin [4], proved the theorem which determines the form of idempotent measures on locally compact abelian groups. As his original proof as well as its version by Rudin [5] is rather complicated, we will give, in this paper, a simple proof of the theorem in the form below; here a bounded measure $\mu$ on a locally compact abelian group $G$ is said to be *canonical* if $\mu$ satisfies the following conditions: The Fourier transform $\hat{\mu}$ of $\mu$ is an integral valued function and $\mu$ is absolutely continuous with respect to the Haar measure of a certain compact subgroup of $G$, namely $\mu = \sum_{i=1}^{n} n_i \gamma_i \cdot m$ where $n_i$ is integer, $\gamma_i$ a continuous character, and $m$ the Haar measure of a compact subgroup.

THEOREM. *Every bounded measure on G with integral valued Fourier transform is a sum of a finite number of mutually orthogonal canonical measures.*

As Rudin showed in [4], it is sufficient to prove the theorem for compact $G$. So, hereafter, let $G$ be a compact abelian group.

Our proof is based on the following facts, (i) and (ii). We consider a set $A$ of measures of the form $\gamma \cdot \mu$ where $\gamma$ ranges over a set of continuous characters on $G$ and $\mu$ is a fixed bounded measure on $G$. Let $\nu$ be an accumulation point of $A$ in the weak*-topology[1] on the space $M(G)$ of bounded measures on $G$, then we have:

(i) For any compact subgroup $G_0$ of $G$, $\nu$, inside $G_0$, either coincides with some $\gamma \cdot \mu$ in $A$ or is singular with respect to the Haar measure of $G_0$;

(ii) $\|\nu\| < \|\mu\|$, if $\hat{\mu}$ is integral valued and $\mu \neq 0$.

(i) can be proved as follows: The restriction of $\nu$ to $G_0$ either coincides with a restriction of some $\gamma \cdot \mu$ in $A$ to $G_0$ or is also an accumulation point of the restrictions of the $\gamma \cdot \mu$ to $G_0$ in the weak*-topology of $M(G_0)$. In the latter case, applying Helson's lemma[2] to the group $G_0$, the restriction of $\nu$ to $G_0$ is singular with respect to the Haar measure of $G_0$.

[1] The weak topology on $M(G)$ as dual of $C(G)$.

[2] Helson's lemma [3] is an immediate consequence of the fact that the Fourier transforms of summable functions on a compact group vanish at infinity, cf. Rudin [5, Lemma 3.5.1, p. 66].

To prove (ii), take an arbitrary positive number $\kappa<\|\nu\|/\|\mu\|$; then we can find $f\in C(G)$ such that $\|f\|_\infty\leqq 1$ and

$$\int_G f\,d\nu > \kappa\|\mu\|.$$

Since the set of all measures $\sigma$ satisfying $\mathcal{R}\int_G f\,d\sigma>\kappa\|\mu\|$ is open and $\nu$ is an accumulation point of $\gamma\cdot\mu$'s, there exist $\gamma_1$, $\gamma_2$ such that $\gamma_1\cdot\mu\neq\gamma_2\cdot\mu$ and the real part of $\int_G f\gamma_1\,d\mu$ and $\int_G f\gamma_2\,d\mu$ are both greater than $\kappa\|\mu\|$. Take $\theta$ so that $d\mu=\theta d|\mu|$, write $f\gamma_j\theta=g_j+ih_j$ $(j=1, 2)$. Then

$$\int_G g_j\,d|\mu| > \kappa\|\mu\|,$$

and hence (cf. Cohen [1, p. 206])

$$\int_G |h_j|\;d|\mu| = \mathcal{I}\left\{\int_G (g_j+i|h_j|)\,d|\mu|\right\} \leqq \sqrt{(\|\mu\|^2-\kappa^2\|\mu\|^2)},$$

so that

$$\begin{aligned}\int_G |1-f\gamma_j\theta|\;d|\mu| &\leqq \int_G (1-g_j)\,d|\mu| + \int_G |h_j|\;d|\mu| \\ &\leqq (1-\kappa)\|\mu\| + \sqrt{(1-\kappa^2)}\|\mu\|,\end{aligned}$$

and so

$$\int_G |\gamma_j - f\gamma_1\gamma_2\theta|\;d|\mu| \leqq (1-\kappa+\sqrt{(1-\kappa^2)})\|\mu\|.$$

Hence it follows that

$$\|\gamma_1\cdot\mu-\gamma_2\cdot\mu\| = \int_G |\gamma_1-\gamma_2|\;d|\mu| \leqq 2(1-\kappa+\sqrt{(1-\kappa^2)})\|\mu\|.$$

On the other hand, since the Fourier transform of $\gamma_1\cdot\mu-\gamma_2\cdot\mu$ is integral valued, we have $\|\gamma_1\cdot\mu-\gamma_2\cdot\mu\|\geqq 1$ and hence we can derive an inequality

$$\kappa < 1-\frac{1}{16\|\mu\|^2};$$

this proves a strengthened form of (ii), namely,

$$\|\nu\| \leqq \|\mu\| - \frac{1}{16\|\mu\|}.$$

THE PROOF OF THE THEOREM. Let $\mu$ be a bounded measure with integral valued Fourier transform and $A$ the set of those $\gamma\cdot\mu$ for which $\int_G\gamma\, d\mu\neq 0$. The closure $\bar{A}$ of $A$ in the weak*-topology is a compact set and does not contain 0, since all $\int_G\gamma\, d\mu$ are integers different from 0. Since the norm is a lower semicontinuous function in the weak*-topology, the norms of the elements in $\bar{A}$ attain the minimum value, say, at $\nu\neq 0$. If $\int_G\gamma\, d\nu\neq 0$, then $\gamma\cdot\nu$ lies in $\bar{A}$, so the set of all $\gamma\cdot\nu$ with $\int_G\gamma\, d\nu\neq 0$ can not have any accumulation point, since such a point, if it exists, must be in $\bar{A}$ and according to (ii), must have norm less than $\|\nu\|$. So this set of $\gamma\cdot\nu$ is finite and we can see easily that such a measure $\nu$ vanishes outside some compact group $G_0$ and is absolutely continuous with respect to the Haar measure of $G_0$, in other words, $\nu$ is canonical.

If $\nu$ is not an accumulation point of $A$, then $\nu$ coincides with some $\gamma\cdot\mu$ and hence $\mu$ itself is canonical. If $\nu$ is an accumulation point of $A$, then, by (i), $\nu$, being not singular with respect to the Haar measure of $G_0$, must coincide with some $\gamma\cdot\mu$ inside $G_0$. In the latter case, the restriction of this $\gamma\cdot\mu$ to $G_0$ is canonical and hence $\mu$ has the same property. So we have a canonical measure $\mu_1=\chi_{G_0}\cdot\mu$[3] and an orthogonal decomposition

$$\mu = \mu_1 + (\mu - \mu_1).$$

The same argument is applicable to $\mu-\mu_1$ and, since its norm decreases at least 1 from that of $\mu$, we can attain finally the desired decomposition.

## REFERENCES

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3. ———, *On a theorem of Szegö*, Proc. Amer. Math. Soc. **6** (1955), 235–242.

4. W. Rudin, *Idempotent measures on abelian groups*, Pacific J. Math. **9** (1959), 195–209.

5. ———, *Fourier analysis on groups*, Interscience, New York, 1962.

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[3] $\chi_{G_0}$ denotes the characteristic function of the set $G_0$.