# A PRIORI ESTIMATES IN SEVERAL COMPLEX VARIABLES[1]

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Classically there are two points of view in the study of global existence problems in the theory of functions of a complex variable. One is to piece together local solutions (such as power series), always staying within the category of holomorphic functions. This method seems to have been initiated by Weierstrass; in the theory of several complex variables it has been implemented by the study of cohomology with coefficients in the sheaf of germs of holomorphic functions (more generally in the sheaf of germs of holomorphic sections of vector bundles). The second approach is to view the Cauchy-Riemann equations as a linear operator on $C^\infty$ functions and to study this operator as an operator in Hilbert space; which leads to the Dirichlet integral and this method was first exploited by Riemann. In the theory of several complex variables this approach has led to the theory of harmonic integrals, which have been developed and widely applied in the compact case and which have recently been extended to the noncompact case. It is this extension which is the main concern of the present lecture. For simplicity we will deal with functions on a domain $M \subset \mathbf{C}^n$, although the results carry over to forms with coefficients in holomorphic vector bundles on finite manifolds.

Let $z^1, \cdots, z^n$ be coordinates in $\mathbf{C}^n$ and let $x^k = \operatorname{Re}(z^k)$ and $y^k = \operatorname{Im}(z^k)$. Then if $u$ is a differentiable function we define $u_{z^k}$ and $u_{\bar{z}^k}$ by

$$u_{z^k} = \frac{1}{2}\left(\frac{\partial u}{\partial x^k} - \sqrt{(-1)}\,\frac{\partial u}{\partial y^k}\right)$$

and

$$u_{\bar{z}^k} = \frac{1}{2}\left(\frac{\partial u}{\partial x^k} + \sqrt{(-1)}\,\frac{\partial u}{\partial y^k}\right).$$

Thus a function is holomorphic if and only if $u_{\bar{z}^k} = 0$, $k = 1, \cdots, n$. Here we are concerned with inhomogeneous equations:

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$$u_{\bar{z}^k} = \alpha_k, \qquad k = 1, \cdots, n; \tag{1}$$

or to use the notation of differential forms

$$\bar{\partial} u = \alpha, \tag{1'}$$

where $\bar{\partial} u = \sum u_{\bar{z}^k} d\bar{z}^k$ and $\alpha = \sum \alpha_k d\bar{z}^k$. Roughly speaking, we ask the following question about the system (1): *Given a "good" $\alpha$ is there a "good" $u$ satisfying the above equations?*

Before considering this question we will illustrate how an affirmative answer can be used to prove the existence of global holomorphic functions. Our example is the Levi problem; which is: given $P \in bM$ ($bM$ denotes the boundary of $M$) to find a holomorphic function $h$ which cannot be continued past $P$ (that is, there is no holomorphic function defined in a neighborhood $U$ of $P$ which equals $h$ in $U \cap M$). In one variable this problem is always trivial, a solution being $(z-P)^{-1}$; however, in several variables the problem is far from trivial and, in fact, it has a solution only under special circumstances. For example, if $M$ is the region between two spheres, one contained in the other, then every holomorphic function on $M$ has (by Hartogs' theorem) an extension to the interior of the larger sphere and hence if $P$ is a point on the smaller sphere then every holomorphic function can be continued past $P$. Suppose that $r$ is a real-valued $C^\infty$ function defined in a neighborhood of $bM$ such that $dr \neq 0$, $r(Q) < 0$ when $Q \in M$ and $r(Q) > 0$ when $Q \notin \bar{M}$, then if $P \in bM$ the *Levi form* at $P$ is the hermitian form

$$\sum r_{z^i \bar{z}^j}(P) a^i \bar{a}^j, \tag{2}$$

acting on $n$-tuples $(a^1, \cdots, a^n)$ which satisfy the equation

$$\sum r_{z^i}(P) a^i = 0. \tag{3}$$

We remark that the numbers of positive and negative eigenvalues of the Levi form are independent of the choice of the function $r$ and of the coordinate system. If all the eigenvalues of the Levi form at $P$ are positive then there exists a neighborhood $U$ of $P$ and a polynomial $w$ such that: the only point in $U \cap \bar{M}$ at which $w$ is zero is $P$. This theorem gives a local solution to the Levi problem; namely, the function $w^{-1}$ is holomorphic in $U \cap M$, but cannot be continued past $P$. Observe that $w^{-1}$ does not in general give a global solution since it may be infinite at some points in $M$. Now let $\rho$ be a $C^\infty$ function with support $U$ which is 1 in a neighborhood of $P$ and let $\alpha = \bar{\partial}(\rho w^{-1})$. Then $\alpha$ is "good" in the sense that it is $C^\infty$ on $\bar{M}$ (in fact it is zero in a neighborhood of $P$). Now suppose that there exists a "good" $u$ such that $\bar{\partial} u=\alpha$, so that in particular $u$ is defined at $P$, then we obtain a global solution to the Levi problem $h$ by setting $h=u-\rho w^{-1}$. Clearly $\bar{\partial} h=0$ and $h$ cannot be continued past $P$.

Our first observation about the system (1) is that since $u_{\bar{z}^k\bar{z}^m}=u_{\bar{z}^m\bar{z}^k}$ we conclude that the components of $\alpha$ satisfy the equation

$$\alpha_{k\bar{z}^m}-\alpha_{m\bar{z}^k}=0$$

whenever there exists $u$ that satisfies (1). We denote $(C^\infty(\overline{M}))^p$ the space of $p$-tuples of complex-valued $C^\infty$ functions on $\overline{M}$, we set $\mathfrak{A}^0=C^\infty(\overline{M})$, $\mathfrak{A}^1=(C^\infty(\overline{M}))^n$ and $\mathfrak{A}^2=(C^\infty(\overline{M}))^{n(n-1)/2}$. If $\theta\in\mathfrak{A}^2$ we index its components by ordered pairs of integers, i.e., $\theta=(\theta_{km})$, where $1\leqq k<m\leqq n$; $\theta$ can also be considered as the 2-form $\theta=\sum\theta_{km}d\bar{z}^k d\bar{z}^m$. Now we define the operator $S$: $\mathfrak{A}^1\to\mathfrak{A}^2$ by

$$S\phi=\begin{cases}\phi_{k\bar{z}^m}-\phi_{m\bar{z}^k} & \text{if } k<m,\\ -\phi_{k\bar{z}^m}+\phi_{m\bar{z}^k} & \text{if } k>m.\end{cases}\tag{4}$$

So that $S\alpha=0$ is a necessary condition for the existence of a solution $u$ of (1). On $(C^\infty(\overline{M}))^p$ we define the $L_2$-inner product and norm by:

$$(\theta, u)=\sum\int_M \theta_i\bar{u}_i dV \quad\text{and}\quad \|\theta\|^2=(\theta,\theta),\tag{5}$$

where $dV$ denotes the volume element. In terms of this inner product the condition $S\alpha=0$ can be expressed by requiring $(\alpha, S^*\theta)=0$ for all $\theta\in\mathfrak{B}^2$, where $S^*$ is the Hilbert space adjoint of $S$ and $\mathfrak{B}^2$ is the intersection of $\mathfrak{A}^2$ with the domain of $S^*$. For the above application it would suffice if we knew that the following orthogonal decomposition holds:

$$\mathfrak{A}^1=\bar{\partial}\mathfrak{A}^0\oplus S^*\mathfrak{B}^2,\tag{6}$$

which is equivalent to the statement that (1) has a solution if and only if $S\alpha=0$.

We will now indicate briefly how (6) is obtained. We denote by $\bar{\partial}^*$ the Hilbert space adjoint of $\bar{\partial}$ and by $\mathfrak{B}^1$ the intersection of $\mathfrak{A}^1$ and the domain of $\bar{\partial}^*$.

*The basic estimate.* Suppose that $\overline{M}$ is compact and that it has smooth boundary given by a function $r$ as above; and that, for every $P\in bM$ the Levi form at $P$ either has all eigenvalues positive or it has at least two negative eigenvalues. Then there exists a constant $C>0$ such that:

$$\sum_{k,m}\|\phi_{k\bar{z}^m}\|^2+\sum_k\int_{bM}|\phi_k|^2 dS\leqq C(\|\bar{\partial}^*\phi\|^2+\|S\phi\|^2)\tag{7}$$

for all $\phi \in \mathcal{B}^1$, where $dS$ is the volume element of $bM$.

Let $Q$ be the hermitian form on $\mathcal{B}^1$ defined by

$$Q(\phi, \psi) = (\bar{\partial}^*\phi, \bar{\partial}^*\psi) + (S\phi, S\psi). \tag{8}$$

Now if $\phi \in \mathcal{B}^1$ and $S\phi \in \mathcal{B}^2$, then

$$Q(\phi, \psi) = (\bar{\partial}\bar{\partial}^*\phi + S^*S\phi, \psi) \tag{9}$$

for all $\psi \in \mathcal{B}^1$. We remark that $Q(\phi, \phi)$ corresponds to the classical Dirichlet integral. The basic estimate is used to prove that: given $\alpha \in \mathcal{A}^1$ there exists $\phi \in \mathcal{B}^1$ such that:

$$Q(\phi, \psi) = (\alpha, \psi) \tag{10}$$

and this implies that $S\phi \in \mathcal{B}^2$ and

$$\bar{\partial}\bar{\partial}^*\phi + S^*S\phi = \alpha. \tag{11}$$

It should be observed that the above equation, in terms of coordinates, gives:

$$\sum_j \phi_{kz^j\bar{z}^j} = \alpha_k, \qquad k = 1, \cdots, n.$$

Now if for any $\alpha \in \mathcal{A}^1$ there exists $\phi$ as above, then we obtain the decomposition:

$$\mathcal{A}^1 = \bar{\partial}\bar{\partial}^*\mathcal{C} \oplus S^*S\mathcal{C}, \tag{12}$$

where $\mathcal{C} = \{\phi \in \mathcal{B}^1 \mid S\phi \in \mathcal{B}^2\}$. Clearly (12) implies the decomposition (6). Thus our problem is reduced to solving (10).

To solve (10) we first show that the basic estimate implies certain "a priori" estimates for (10). By an "a priori" estimate is meant a bound on some norm applied to the solution $\phi$ in terms of $\alpha$; that is, bounds on the assumption that a solution exists. If the basic estimate holds then for each integer $s \geqq 0$ there exists $C_s > 0$ such that:

$$\|\phi\| \leqq C_0\|\alpha\| \quad \text{and} \quad \|\phi\|_s \leqq C_s\|\alpha\|_{s-1} \quad \text{if} \quad s \geqq 1, \tag{13}$$

for all $\phi \in \mathcal{B}^1$ with $S\phi \in \mathcal{B}^2$, where $\alpha$ is given by (11). The norms $\| \ \|_s$ are defined by:

$$\|\phi\|_s^2 = \sum_{j_1+\cdots+j_n \leq s} \sum_k \| D_1^{j_1} \cdots D_n^{j_n} \phi_k \|^2, \tag{14}$$

where

$$D_i = \begin{cases} \partial/\partial x^i & \text{if } 1 \leqq i \leqq n, \\ \partial/\partial y^{i-n} & \text{if } n < i \leqq 2n. \end{cases}$$

The estimates (13) show that $\phi$ "gains" one derivative. In the standard theory for boundary value problems of second order elliptic systems the solution "gains" two derivatives, i.e., one obtains estimates of the form $\|\phi\|_s \leqq \text{const } \|\alpha\|_{s-2}$; such estimates are called coercive, they have been extensively studied and they imply an existence theorem of the type that we require. In the case of our problem it can be shown that coercive estimates do not hold; nevertheless, the problem can be "approximated" by regular problems for which coercive estimates hold and the required existence theorem is obtained by taking a limit of the solutions of the approximating problems. The approximating problems correspond to the forms $Q_\epsilon$ defined by:

$$Q_\epsilon(\phi, \psi) = Q(\phi, \psi) + \epsilon \sum_{i,k} (D_i\phi_k, D_i\psi_k) \tag{15}$$

for $\epsilon > 0$ and $\phi, \psi \in \mathcal{B}^1$. Then given $\alpha \in \mathcal{A}^1$ there exists a unique $\phi_\epsilon \in \mathcal{B}^1$ such that:

$$Q_\epsilon(\phi_\epsilon, \psi) = (\alpha, \psi) \tag{16}$$

for all $\psi \in \mathcal{B}^1$. Furthermore the $\phi_\epsilon$ satisfy the estimates (13) with constants that are independent of $\epsilon$. It then follows that there is a sequence $\{\epsilon_\nu\} \to 0$ such that $\{\phi_{\epsilon_\nu}\}$ is a Cauchy sequence in the norm $\| \ \|_s$ for every $s$, hence the limit $\phi$ is in $\mathcal{A}^1$ and gives the required solution.

To conclude we will give the proof of the basic estimate in a special case.

*Proof of the basic estimate when the Levi form is positive definite.* First, by integration by parts we have

$$(\phi, \bar{\partial} u) = \left(-\sum \phi_{kz^k}, u\right) + \sum \int_{bM} r_{z^k}\phi_k \bar{u}\, dS$$

so that if $\phi \in \mathcal{B}^1$ we have

$$\sum r_{z^k}\phi_k = 0 \text{ on } bM \tag{17}$$

and

$$\bar{\partial}^*\phi = -\sum \phi_{kz^k}. \tag{18}$$

Since $\sum r_{z^k}\phi_k$ vanishes on $bM$ then at each point of $bM$ its gradient equals $\lambda dr$, where $\lambda$ is a function on $bM$. Thus differentiating with respect to $\bar{z}^m$ we obtain

$$\sum_k r_{z^k\bar{z}^m}\phi_k + \sum_k r_{z^k}\phi_{k\bar{z}^m} = \lambda r_{\bar{z}^m} \text{ on } bM,$$

then multiplying by $\bar{\phi}_m$, summing with respect to $m$ and applying the conjugate of (17) to the right side, we have:

$$\sum_{k,m} r_{z^k\bar{z}^m}\phi_k\bar{\phi}_m + \sum_{k,m} r_{z^k}\phi_{k\bar{z}^m}\bar{\phi}_m = 0$$

and since the Levi form is positive definite there exists a constant such that

$$-\operatorname{Re}\sum_{k,m} r_{z^k}\phi_{k\bar{z}^m}\bar{\phi}_m \geqq \text{const} \sum |\phi_k|^2. \tag{19}$$

Now

$$\begin{aligned} \|S\phi\|^2 &= \sum_{k<m} \|\phi_{k\bar{z}^m} - \phi_{m\bar{z}^k}\|^2 \\ &= \sum_{k,m} \|\phi_{k\bar{z}^m}\|^2 - \sum_{k,m} (\phi_{k\bar{z}^m}, \phi_{m\bar{z}^k}) \end{aligned} \tag{20}$$

and integrating by parts twice, we have:

$$\sum (\phi_{k\bar{z}^m}, \phi_{m\bar{z}^k}) = \sum (\phi_{kz^k}, \phi_{mz^m}) - \sum \int_{bM} \phi_{kz^k} r_{\bar{z}^m}\bar{\phi}_m dS + \sum \int_{bM} r_{z^k}\phi_{k\bar{z}^m}\bar{\phi}_m dS.$$

Now by (18) the first term on the right equals $\|\bar{\partial}^*\phi\|^2$ and by (17) the second term vanishes. The desired estimate is then obtained by substituting this into (20) and applying (19).

Our purpose in the preceding is to introduce the reader to some of the ideas used in this approach to several complex variables, we have left bibliographical remarks to the end. Here we give a representative selection of the recent articles on the topics discussed above. The Levi problem for domains over $C^2$ was first solved by K. Oka (see *Domaines d'holomorphie*, J. Sci. Hiroshima Univ. Ser. A **7** (1937), 115–130). This solution was generalized independently by H. Bremermann, F. Norguet and K. Oka in articles that appeared in 1954. A solution based on differential equations was outlined by L. Ehrenpreis in *Some applications of the theory of distributions to several complex variables*, Conference on Analytic Functions (1957), 65–79. The problem for manifolds was solved, using methods of sheaf theory by H. Grauert in *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472.

A variant of the problem described here was first formulated by P. R. Garabedian and D. C. Spencer in *Complex boundary value problems*, Trans. Amer. Math. Soc. **73** (1952), 223–242. The basic estimate was established in a special case by C. B. Morrey in *The analytic embedding of abstract real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 159–201. The complete solution of the $\bar{\partial}$-Neumann problem (that is the boundary problem discussed here) for strongly pseudo-convex manifolds (i.e. when the Levi form is positive definite) was obtained by the author (see *Harmonic integrals on strongly pseudo-convex manifolds*. I, Ann. of Math. (2) **78** (1963), 112–148; II, ibid. **79** (1964), 450–472). Those papers also include various applications including the new solution of the Levi problem outlined here. The method of approximation by coercive problem is explained in *Non-coercive boundary value problems*, J. J. Kohn and L. Nirenberg, J. Pure Appl. Math. (to appear). The extension of the method to the case of Levi forms with some negative eigenvalues and other generalizations and applications were obtained by L. Hörmander in *Existence theorems for the $\bar{\partial}$ operator by $L^2$ methods*, (to appear). Some of these results were previously conjectured in the above-mentioned paper by Enrenpreis and proved by A. Andreotti and H. Grauert in *Thêorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

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