INJECTIVE ENVELOPES OF BANACH SPACES ARE RIGIDLY ATTACHED

BY J. R. ISBELL[1]

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Introduction. H. B. Cohen has constructed [2] an injective (linear) envelope $\epsilon_l E$ for every real or complex Banach space E, and shown that it is unique up to a linear isometry. The real case of Cohen's result provides most of the answer to a question I had asked, which concerned the injective metric envelope $\epsilon_m E$. It is known that in real Banach spaces the metric and metric linear notions of "injective" coincide [1], [6]. The question whether the metric space $\epsilon_m E$ and the Banach space $\epsilon_l E$ coincide is unambiguous, at least in the sense that the metric structure of a real Banach space determines the linear structure [5]. The answer:

THEOREM 1. *For a real Banach space E, $\epsilon_l E = \epsilon_m E$.*

I cannot deduce this from Cohen's results on ϵ_l, but I get it from my results on ϵ_m [4]. Either approach yields the real case of

THEOREM 2. *A linear autoisometry of a Banach space E can be extended in only one way to a linear autoisometry of $\epsilon_l E$.*

However, the proof in the manner of Cohen covers both cases and is shorter.

Each approach proves a strengthened form of Theorem 2. By a lemma of Cohen [2], any other linear extension has norm > 1. In the real case, any nonlinear extension increases some distance.

Proofs. Let us do Theorem 2 first. It suffices to show that the subspace of all points of $\epsilon_l E$ left fixed by a linear autoisometry T different from the identity lies in an injective proper subspace. Now the form of $\epsilon_l E$ is known (Nachbin-Goodner-Kelley-Hasumi Theorem, re-proved in [2]); it is the space $C(X)$ of all continuous scalar-valued functions on an arbitrary extremally disconnected compact space. The form of T is easy to determine; it must consist of composition with an autohomeomorphism τ of X and multiplication by a continuous function t on X to the scalars of absolute value 1. (This is readily deduced from the characterization of the extreme points of $C(X)^*$ [3].)

Since $T \neq 1$, either $\tau \neq 1$ or $\tau = 1$ but $t \neq 1$. In the former case there

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is a nonempty open-closed set U of X disjoint from $\tau(U)$, and the fixed points of T lie in the subspace of all f satisfying $t(u)f(\tau(u))=f(u)$ for u in U. In the latter case there is a nonempty open-closed set U of X on which $t-1$ has no zero, and the fixed points of T must vanish on U. In either case an injective proper subspace $C(X-U)$ contains the fixed points of T. Theorem 2 is proved.

For the linear strengthening, Cohen's lemma [2] says that if an extension $T\colon \epsilon_l E\to Y$ of an isometric embedding $E\to Y$ has norm 1, it is an isometric embedding. The image is then injective; in the present case, this means T is an autoisometry.

Theorem 1 requires some details of the construction [4] of $\epsilon_m X$. A real-valued function f on a metric space X is called *extremal* if it is pointwise minimal subject to $f(x)+f(y)\geqq d(x, y)$. The extremal functions on X form a metric space Y with the distance $d(f, g)=\sup|f(x)-g(x)|$.

Defining $e\colon X\to Y$ by $e(x)(x')=d(x, x')$, one has an injective envelope. A uniqueness theorem is given in [4], but it can be strengthened. The distance from $f\in Y$ to $e(x)$ is $f(x)$ [4]. Thus the isometry connecting two envelopes $e\colon X\to Y$, $e'\colon X\to Y'$ is unique. Since the functions are minimal, one has a strong version of Theorem 2: every extension over $\epsilon_m X$ of an autoisometry of X, except one extension (which is an autoisometry), increases some distance. Note also: every similitude of X can be extended uniquely to a similitude of $\epsilon_m X$.

Further remarks: since a subspace Z of $\epsilon_m X$ containing X has the same injective envelope, the extremal functions on Z are precisely the unique extensions of extremal functions on X. Z is injective if and only if every extremal function on Z has a zero.

For Theorem 1, it suffices to exhibit an isometry of $\epsilon_m E$ with a Banach space taking E to a linear subspace. We begin with the well-known result [7] that E can be linearly isometrically embedded in some injective Banach space I.

Consider the subspaces F of I containing E such that an injective metric envelope $e\colon E\to Y$ can be extended to an isometric embedding $f\colon F\to Y$, partially ordered by inclusion. As injective metric spaces are complete [1], and the mappings f are unique, Zorn's Lemma applies: there is a maximal such subspace, Z. We are done if every extremal function on Z has a zero. Suppose the contrary; let g be a positive extremal function on Z. Considering Z as a subspace of its injective envelope Y, extend the inclusion $i\colon Z\to I$ over Y. g determines a point of Y, mapped to a point p of I. For every z in Z, $d(p, z)\leqq g(z)$. Since $d(p, x)+d(p, y)\geqq d(x, y)$ and g is extremal, $d(p, z)$ is $g(z)$. For every element $q=\alpha p+x$ of the subspace G generated by Z

and p, the function f_q given by $d(q, z)$ on Z is extremal; it is just g, transformed by the similitude $z \to \alpha z$ and the isometry $z \to z + x$. Moreover, the distance between any two points q, r of G is just $\sup|f_q - f_r|$; this is obvious if the line joining q and r meets Z, and when it is parallel one can approximate the segment qr by segments of lines meeting Z. Hence $e: E \to Y$ extends to an isometric embedding of G in Y, contradicting the maximality of Z. Theorem 1 is proved.

Concluding remarks. A one-dimensional complex Banach space, though linearly injective, is not metrically injective, and one easily deduces that none of them are metrically injective. For a complex Banach space E and the real Banach space $\epsilon_m E$, the operations of complex scalars, as a semigroup of similitudes of E, extend over $\epsilon_m E$. The extension must fail to preserve the additive structure of the scalars.

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