$$\|F\|_{\mathfrak{F}l^q} \leqq 2K_pN^{1/q-1}\gamma^{1/p-1} = 2K_p\epsilon^{1/q-1}\cdot\gamma^{1/p-1/q}$$

so that, if N is large enough $\|F\|_{\mathfrak{F}l^q} < \epsilon$. Put

$$E_{\epsilon,p} = \{x;\, F(x) = 1\}.$$

Now $F(x)=1 \Leftrightarrow f_\gamma(\lambda_j x)=1$ for all j, so that measure $(E_{\epsilon,p}) \geqq 2\pi - N\gamma = 2\pi - \epsilon$.

Let μ be carried by $E_{\epsilon,p}$, $\mu \in \mathfrak{F}l^p$, then $|\hat{\mu}(n)| = |\int e^{-inx}d\mu| = |\int e^{-inx}F(x)d\mu| = |\sum \hat{F}(m-n)\hat{\mu}(m)| \leqq \|F\|_{\mathfrak{F}l^q}\|\mu\|_{\mathfrak{F}l^p} < \epsilon\|\mu\|_{\mathfrak{F}l^p}$ which proves the lemma.

THEOREM. *There exists a set E of positive measure on T which is a set of uniqueness for $\bigcup_{p<2} l^p$.*

PROOF. Take $\epsilon_n = 10^{-n}$, $p_n = 2 - \epsilon_n$,

$$E = \bigcap_{n=36}^{\infty} E_{\epsilon_n, p_n}.$$

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INJECTIVE ENVELOPES OF BANACH SPACES

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1. **Introduction.** We consider the category whose objects are Banach spaces and whose *maps* are the linear operators of norm not exceeding 1 from one Banach space into another. A Banach space Z is *injective* if it has the same Hahn-Banach extension property that is possessed by the scalars (real or complex); that is, any Z-valued map from a subspace of a Banach space Y extends to a Z-valued map of the same norm on all of Y. An injective envelope of a Banach space B is a pair $(I, \epsilon B)$, ϵB an injective Banach space and I: $B \rightarrow \epsilon B$ a linear isometry (our linear isometries need not be onto), such that the only subspace of ϵB that is injective and contains $I[B]$ is ϵB itself. In this note, we demonstrate the existence and uniqueness of the injective envelope of a Banach space and, in the process, we give a short proof of the fact that an injective Banach space is linearly isometric with a function space $C(M)$, M compact Hausdorff and extremally disconnected.

For X a compact Hausdorff space, $C(X)$ denotes the Banach space of all continuous scalar-valued functions on X with the sup norm.

A topological space is *extremally disconnected* if the closure of every open subset is open; a continuous function is *minimal* if it is onto, but no longer onto when restricted to an arbitrary closed proper subset of its domain. Fundamental to our construction is Gleason's result [1]: for any compact Hausdorff space X, there is a minimal continuous function $i: M \to X$ with M compact Hausdorff and extremally disconnected. The writer wishes to thank J. Isbell for suggesting the problem of determining the injective envelope of a Banach space; Isbell conjectured that the injective envelope of a function space $C(X)$ would be $(I, C(M))$ where I is the linear isometry induced by Gleason's function $i: M \to X$.

THEOREM 1 (NACHBIN-GOODNER-HASUMI). *If M is compact Hausdorff and extremally disconnected, then $C(M)$ is injective.*

PROOF. Phillips [6] showed that the Banach space $m(D)$ of bounded scalar-valued functions on a set D is injective; consequently, a function space $C(K)$, K the Stone-Cech compactification of a discrete space, is injective. Given a compact Hausdorff extremally disconnected space M, the combined results of Gleason [1] and Rainwater [7] imply that M is a retract of a suitably chosen Stone-Cech compactification K of a discrete space. It follows that $C(M)$ is a retract of $C(K)$ and therefore injective.

2. **Construction of the envelope.** For any subset Q of a real or complex linear space, EQ denotes the set of extremal points of Q and co$Q$ the convex hull. For any Banach space B, B^* denotes the adjoint space and B^π the unit sphere of B^*; cl$^*Q$ denotes the weak* closure of a subset Q of B^*. The set of all scalars of norm 1 is denoted by C_0.

Let B be a Banach space. Let U and X be subsets of B^π satisfying (1) X is weak* closed; U is contained in $EB^\pi \cap X$. (2) $\text{cl}^*U = X$. (3) $\text{cl}^*C_0U = \text{cl}^*EB^\pi$. (4) For each u in U, $C_0u \cap X = \{u\}$. For B real, such a U and X are easily constructed as follows. Let W be a subset of cl$^*EB^\pi$ maximal with respect to being both open in cl$^*EB^\pi$ and disjoint from $-W$. Then $W \cup -W$ is weak* dense in cl$^*EB^\pi$; hence, so is $(EB^\pi \cap W) \cup -(EB^\pi \cap W)$. Let $U = EB^\pi \cap W$ and $X = \text{cl}^*U$.

For B complex, proceed as follows. Call a subset W of B^* *deleted* if $w \in W$ implies $kw \in W$ for all but exactly one k in C_0, and say that W is *circled* if $C_0W \subset W$. If W is circled, so is cl$^*W$; in particular, cl$^*EB^\pi$ is circled. *Every nonvoid open circled subset W of* cl$^*EB^\pi$ *contains a nonvoid open deleted subset.* For choose w in W and b in B such that $w(b) \in D$, the open unit disc in the plane with the interval $[0, 1)$ removed. If $J(b)$ is the weak* continuous functional determined by b, i.e., $J(b)(z) = z(b)$ for all z in B^*, then $J(b)^{-1}(D) \cap W$ is the required set. Using Zorn's lemma, let W be a subset of $\mathrm{cl}^* EB^\pi$ maximal with respect to being both open in $\mathrm{cl}^* EB^\pi$ and deleted. Then $C_0 W$ is dense in $\mathrm{cl}^* EB^\pi$. Since $C_0 W$ is open and dense in $\mathrm{cl}^* EB^\pi$, $EB^\pi \cap C_0 W$ is dense there, too. Let U denote the set of all u in EB^π such that $u \notin W$ but $ku \in W$ for all k in C_0, $k \neq 1$. Define $X = \mathrm{cl}^* U$. Thus (1) and (2) are satisfied trivially, and (3) follows from the equality $C_0 U = EB^\pi \cap C_0 W$. The fact that W and X are disjoint yields (4).

With U and X thus chosen, X is a compact Hausdorff space; let i: $M \to X$ be Gleason's function. Let I: $B \to C(M)$ be defined by $I(b)(m) = i(m)(b)$. The continuity of I is used to show that I is well defined, I is obviously linear, and I is a map because X is contained in B^π. To show I is an isometry, let $b \in B$ and choose $v \in EB^\pi$ such that $\|b\| = |v(b)|$. Using (3), let v_α be a net in $C_0 U$ converging to v in the weak* topology; hence, $|v_\alpha(b)|$ converges to $\|b\|$. Given $\epsilon > 0$, choose α such that $\|b\| - \epsilon < |v_\alpha(b)|$. For a unique k in C_0, $v_\alpha = ku$, u in U. Thus $\|b\| - \epsilon < |u(b)| = |i(m')(b)| = |I(b)(m')| \leqq \|I(b)\|$ for a suitable $m' \in M$. Since $\epsilon > 0$ was arbitrary, $\|b\| \leqq \|I(b)\|$.

THEOREM 2. *The pair $(I, C(M))$ is the essentially unique injective envelope of B; indeed, given a Banach space Y, a linear isometry G: $B \to Y$, and any map H: $C(M) \to Y$ such that $H \circ I = G$, then H is a linear isometry.*

PROOF. We first prove H is a linear isometry; note, $\|H\| \leqq 1$ by assumption. Let f belong to $C(M)$ and $\epsilon > 0$ be given. Assume that f takes on the value $\|f\|$, no restriction since we are interested in the norm of f and can therefore work with kf for any k in C_0. Set $M(\epsilon)$ equal to the set of all m in M such that $f(m)$ lies in $D(\epsilon)$, the open disc in the plane of radius ϵ centered at $\|f\|$. Because i is minimal, $i[M \backslash M(\epsilon)]$ is a proper closed subset of X. By the fact (2) that U is dense in X, let u belong to U and to $X \backslash i[M \backslash M(\epsilon)]$; consequently, $i^{-1}(u) \subset M(\epsilon)$. Letting e: $M \to C(M)^*$ denote the homeomorphism $e(m)(g) = g(m)$, $e[i^{-1}(u)]$ is a collection of functionals whose values at f lie in $D(\epsilon)$. Consequently,

(a) if z is in $\mathrm{cl}^* \mathrm{co}\, e[i^{-1}(u)]$, then $\|f\| - \epsilon \leqq |z(f)|$.

Next, suppose z is in $EC(M)^\pi \cap I^{*-1}(u)$. Then $z = ke(m)$ for some m in M, k in C_0; and so, $u = I^* z = kI^* e(m) = ki(m)$. Therefore $i(m)$ is a member of $C_0 u \cap X$ so by (4), $i(m) = u$ and $k = 1$. Therefore m is in $i^{-1}(u)$ and $z = e(m)$ is in $e[i^{-1}(u)]$. Taking closed convex hulls:

(b) $$\mathrm{cl}^*\mathrm{co}(EC(M)^\pi \cap I^{*-1}(u)) \subset \mathrm{cl}^*\mathrm{co}\, e[i^{-1}(m)].$$

The Hahn-Banach theorem applied to $u \circ G^{-1}$ yields a y in Y such that $G^*y = u$; hence, $I^*(H^*y) = (H \circ I)^*y = G^*y = u$. Consequently H^*y is in $C(M)^\pi \cap I^{*-1}(u)$ a set contained in $\mathrm{cl}^*\mathrm{co}(EC(M)^\pi \cap I^{*-1}(u))$ (take the weak* closed convex hull of both sides of $E(C(M)^\pi \cap I^{*-1}(u)) \subset EC(M)^\pi \cap I^{*-1}(u)$, reducing the left-hand side of the resulting inclusion by means of the Krein-Milman theorem). Using (b) and then (a), $\|f\| - \epsilon \leqq |H^*y(f)| = |y(H(f))| \leqq \|H(f)\|$. Since ϵ was arbitrary, $\|f\| \leqq \|H(f)\|$. Therefore, H is an isometry.

Suppose Z is a subspace of $C(M)$ containing $I[B]$. If Z is injective, there is a map $H: C(M) \to Z$ such that $H(z) = z$ for all z in Z. Letting $G: B \to Z$ denote the map I with range restricted, $H \circ I = G$ so, by the above, H is an isometry. But the only way that H can be $1-1$ is for Z to be all of $C(M)$. Therefore, $(I, C(M))$ is an injective envelope of B.

This injective envelope is unique in the sense that if (G, Y) is another injective envelope of B, there is a linear isometry $H: C(M) \to Y$ onto such that $H \circ I = G$. For Y injective provides a map $H: C(M) \to Y$ such that $H \circ I = G$, and the lemma implies that H is an isometry. This means that $H[C(M)]$ is an injective Banach space which, as a subspace of Y, contains $G[B]$. Therefore, $H[C(M)] = Y$; i.e., H is onto.

THEOREM 3 (KELLEY-HASUMI). *An injective Banach space B is linearly isometric with a function space $C(M)$, M compact Hausdorff and extremally disconnected.*

PROOF. Suppose B is injective. Then if $I: B \to C(M)$ is the map constructed above, $I[B]$ is injective and therefore equal to $C(M)$.

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