THE TEICHMÜLLER SPACE OF AN ARBITRARY FUCHSIAN GROUP[1]

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1. **Introduction.** Let U be the upper half plane. Let Σ be the set of quasiconformal self-mappings of U which leave 0, 1, and ∞ fixed. The universal Teichmüller space of Bers is the set T of mappings $h: R \rightarrow R$ which are boundary values of mappings in Σ.

Let M be the open unit ball in $L_\infty(U)$. For each μ in M, let f^μ be the unique mapping in Σ which satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z. \tag{1}$$

We map M onto T by sending μ to the boundary mapping of f^μ. T is given the quotient topology induced by the L_∞ topology on M. The right translations, of the form $h \rightarrow h \circ h_0$, are homeomorphisms of T.

We shall also associate to each μ in M a function ϕ^μ holomorphic in the lower half plane U^*. For each μ, let w^μ be the unique quasiconformal mapping of the plane on itself which is conformal in U^*, satisfies (1) in U, and leaves 0, 1, and ∞ fixed. ϕ^μ is the Schwarzian derivative $\{w^\mu, z\}$ of w^μ in U^*. By Nehari [3], ϕ^μ belongs to the Banach space B of holomorphic functions ψ on U^* which satisfy

$$\|\psi\| = \sup |(z - z^*)^2\psi(z)| < \infty.$$

It is known [1, pp. 291–292] that $\phi^\mu = \phi^\nu$ if and only if f^μ and f^ν have the same boundary values. Hence, there is an injection $\theta: T \rightarrow B$ which sends the boundary function of f^μ to ϕ^μ. We shall write $\theta(T) = \Delta$.

Now let G be a Fuchsian group on U; that is, a discontinuous group of conformal self-mappings of U, not necessarily finitely generated. The mapping f in Σ is compatible with G if $f \circ A \circ f^{-1}$ is conformal for every A in G. The Teichmüller space $T(G)$ is the set of h in T which are boundary values of mappings compatible with G. The space $B(G)$ of quadratic differentials is the set of ϕ in B such that

$$\phi(Az)A'(z)^2 = \phi(z) \qquad \text{for all } A \text{ in } G.$$

Ahlfors proved in [1] that Δ is open in B. Bers [2] proved that θ maps T homeomorphically on Δ and maps $T(G)$ onto an open subset of $B(G)$. These results are summed up in the following theorems:

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THEOREM 1. *The mapping* $\mu \to \phi^\mu$ *is continuous.*

THEOREM 2. *The mapping* $\mu \to \phi^\mu$ *is open.*

THEOREM 3. $\theta(T(G))$ *is an open subset of* $B(G)$.

Our purpose here is to give new, more elementary proofs of Theorems 2 and 3. In particular, we notice that Theorem 3 is a straightforward consequence of Theorems 1 and 2 and the lemma in the next section.

2. **The space** $D(G)$. For each Fuchsian group G, we denote by $D(G)$ the set of h in T such that $h \circ A \circ h^{-1}$ is the boundary function of a conformal self-mapping of U for every A in G. Clearly, $T(G)$ is contained in $D(G)$.

LEMMA. $\theta(D(G)) = B(G) \cap \Delta$.

PROOF. For each A in G and ϕ^μ in Δ,

$$\phi^\mu(Az)A'(z)^2 = \{w^\mu, Az\}A'(z)^2 = \{w^\mu \circ A, z\}.$$

Therefore, $\phi^\mu \in B(G) \cap \Delta$ if and only if for each A in G, the restriction of $w^\mu \circ A \circ (w^\mu)^{-1}$ to $w^\mu(U^*)$ is a linear transformation.

Let ϕ^μ belong to $\theta(D(G))$. Let $f = f^\mu$ and $w = w^\mu$. Let g be the conformal map of U onto $w(U)$ such that $w = g \circ f$. For each A in G there is a conformal map A_1: $U \to U$ which agrees with $f \circ A \circ f^{-1}$ on the real axis. We put S equal to $w \circ A \circ w^{-1}$ in $w(U^*)$ and to $g \circ A_1 \circ g^{-1}$ in the closure of $w(U)$. S is quasiconformal everywhere and conformal off $w(R)$. Hence S is everywhere conformal, and $\phi^\mu \in B(G) \cap \Delta$.

Conversely, suppose $\phi^\mu \in B(G) \cap \Delta$. Let $w = w^\mu$, $f = f^\mu$, and $g = w \circ f^{-1}$. Given A in G, let S be the linear transformation which agrees with $w \circ A \circ w^{-1}$ in $w(U^*)$. By continuity, $S \circ w = w \circ A$ on the real axis. Therefore, $f \circ A \circ f^{-1} = g^{-1} \circ S \circ g$ on R, and the boundary function h of f belongs to $D(G)$. But $\theta(h) = \phi^\mu$. Q.E.D.

3. **Proof of Theorem 2.** Let $\phi_0 = \phi^\mu$ be a point of Δ. We must show that every neighborhood of μ covers a neighborhood of ϕ_0. Ahlfors [1] proves that if $\|\phi - \phi_0\|$ is sufficiently small, ϕ belongs to Δ. With Ahlfors, we write $\phi = \{w^\nu, z\}$ where $w^\nu = \hat{f} \circ w^\mu$. It suffices to prove that the complex dilatation of $\hat{f}$ tends to zero with $\|\phi - \phi_0\|$.

According to [1, p. 300], $\hat{f}$ is the limit of a sequence of mappings $\hat{f}_n$. From formula (13) of [1] and the chain rule, we compute that the complex dilatation ρ_n of $\hat{f}_n$ satisfies

$$\|\rho_n\|_\infty < \frac{\|\phi - \phi_0\|}{\delta - \|\phi - \phi_0\|}$$

where δ is a positive constant depending only on μ. Obviously, $\|\rho_n\|_\infty$ tends to zero with $\|\phi-\phi_0\|$. Q.E.D.

4. **Proof of Theorem 3.** We show first that $\theta(T(G))$ contains a neighborhood of the origin in $B(G)$. It is well known [1, pp. 297–299] that every ϕ in B with $\|\phi\|<2$ has the form ϕ^μ for

$$\mu(z) = \tfrac{1}{2}(z - z^*)^2\phi(z^*). \tag{2}$$

Moreover, it is a simple consequence of the chain rule that f^μ is compatible with G if and only if

$$\mu(Az) = \mu(z)A'(z)/A'(z)^* \qquad \text{for all } A \text{ in } G. \tag{3}$$

If $\phi\in B(G)$ and $\|\phi\|<2$, the μ in (2) satisfies (3). Hence, $\theta(T(G))$ contains the open unit ball in $B(G)$.

Now let f^ν be any mapping compatible with G and let G^ν be the Fuchsian group $f^\nu \circ G \circ (f^\nu)^{-1}$. Let α: $T\to T$ be the right translation which carries the boundary mapping of f^ν to the identity. It is obvious that α maps $T(G)$ onto $T(G^\nu)$ and $D(G)$ onto $D(G^\nu)$. Let $\beta:\Delta\to\Delta$ be the homeomorphism $\theta \circ \alpha \circ \theta^{-1}$. By the Lemma, β maps the open set $B(G)\cap\Delta$ in $B(G)$ onto the open set $B(G^\nu)\cap\Delta$ in $B(G^\nu)$. Moreover, β maps ϕ^ν to zero.

We have seen that $\theta(T(G^\nu))$ contains the open unit ball N in $B(G^\nu)$. Since α maps $T(G)$ on $T(G^\nu)$, $\beta^{-1}(N)$ is contained in $\theta(T(G))$. Since β is a homeomorphism of $B(G)\cap\Delta$ on $B(G^\nu)\cap\Delta$, $\beta^{-1}(N)$ is open in $B(G)$. Therefore, $\theta(T(G))$ contains a neighborhood of ϕ^ν in $B(G)$. Since f^ν was any mapping compatible with G, $\theta(T(G))$ is an open set. Q.E.D.

References

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2. L. Bers, *Automorphic forms and general Teichmüller spaces*, Proceedings of the Conference on Complex Analysis, Univ. of Minnesota, 1964 (to appear).

3. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.

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