ON THE UNKNOTTEDNESS OF THE FIXED POINT SET OF DIFFERENTIABLE CIRCLE GROUP ACTIONS ON SPHERES—P. A. SMITH CONJECTURE

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The original P. A. Smith conjecture is that there are no Z_p actions on S^3 with a knotted S^1 as fixed point set. The so-called generalized P. A. Smith conjecture is that there are no Z_p or circle group actions on S^n with a knotted S^{n-2} as fixed point set [2], [8]. Mazur [5], [6] tried to give counterexamples for the cases $n=4$, 5 but there are several mistakes. In this paper, we show that the P. A. Smith conjecture is true for differentiable circle group actions. According to Giffen [3], there are examples of differentiable Z_p actions on S^n, $n\geqq 5$, p arbitrary, with knotted S^{n-2} as their fixed point sets.

In view of the fact that the cohomological theories for Z_p actions and circle group actions are always parallel, it becomes more interesting to find the *differences* between Z_p actions and circle group actions. We will show that the circle group actions are more regular, in a sense, than Z_p actions.

THEOREM I. *Suppose given a differentiable action of S^1 on S^n, $n\neq 4$, with its fixed point set $F=S^{n-2}$, then F is necessarily unknotted. If $n=4$, then S^n-F has the homotopy type of a circle. Actually, except for the cases $n=4$, 5, the following stronger result is true.*

THEOREM I′. *A differentiable action of S^1 on S^n with an $(n-2)$-dimensional fixed point set F is orthogonal if and only if F is an $(n-2)$-sphere.*

The above theorems are just special cases of the following classification theorem. First, we give a construction.

Construction. Given a compact contractible manifold X of dimension $n-1$, $n\geqq 5$, we may have a circle group action on the smoothed $D^2\times X$ simply by letting $g\cdot(y, x)=(g\cdot y, x)$.

By h-cobordism theorem, $D^2\times X$ is a differentiable disc. If we restrict the action to the boundary of $D^2\times X$, we get a circle group action on S^n with its orbit space diffeomorphic to X and its fixed point set, F, diffeomorphic to ∂X.

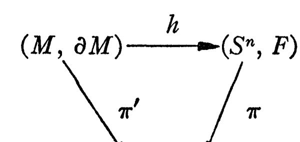
THEOREM II. *For $n\geqq 5$, every differentiable circle group action on S^n with* dim $F=n-2$ *is differentiably equivalent to one and only one of the examples constructed above.*

PROOF. We may assume that the given action is effective; if not, we may consider a quotient group which is again a circle group. Since the group S^1 is abelian, the principal isotropy subgroup is $\{e\}$.

By P. A. Smith theory and the assumption dim $F=n-2$ we see that F is an $(n-2)$-cohomology sphere. By Bochner's theorem, a differentiable action is always orthogonal around a fixed point, x. In our case, the representation is faithful and leaves an $(n-2)$-subspace fixed. It is easy to see that the representation is always the standard one, and hence there exists an invariant neighborhood N of F in S^n such that S^1 acts freely on $N-F$.

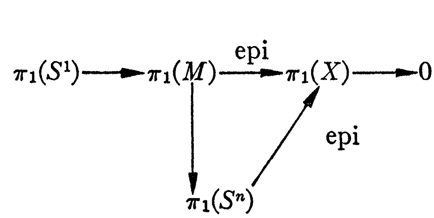
We claim that S^1 acts freely on S^n-F, i.e., there are only two types of isotropy subgroups, namely $\{e\}$ and the whole group S^1. Suppose the contrary, then there exists a Z_p subgroup in S^1, p a suitable prime, with $F(Z_p, S^n)\supset F$, $F(Z_p, S^n)\neq F$. By the above fact that S^1 acts freely on $N-F$, we see that $F(Z_p, S^n)$ has at least two components, which is clearly a contradiction to P. A. Smith theory that $F(Z_p, S^n)$ is a Z_p-homology sphere.

It follows from the fact that S^1 acts freely on S^n-F, that the associated orbit space X is a manifold with boundary $\partial X=\pi(F)$, π the projection map. Moreover, we may blow up S^n along F to get a manifold with boundary $(M, \partial M)$ such that S^1 acts freely on $(M, \partial M)$ and the following diagram is naturally defined and commutative [4]:



where h is an equivariant relative diffeomorphism and π, π' are projections onto their common orbit spaces $(X, \partial X)$.

Since $S^1\to M\to X$ is a fibration, we have



hence, $\pi_1(X)=0$.

A similar argument shows that $H_i(X)=0$ for $i \geqq 1$ and hence X is compact contractible and the fibration

$$S^1 \to M \to X$$

must be trivial, i.e., $(M, \partial M)=(S^1 \times X, S^1 \times \partial X)$. By the construction of $(M, \partial M)$, (S^n, F) may be obtained from $(M, \partial M)=(S^1 \times X, S^1 \times \partial X)$ by identifying every circle $S^1 \times \{x\}$; $x \in \partial X$ to a point, which is equal to $(\partial(D^2 \times X), \{0\} \times \partial X)$ up to diffeomorphism. q.e.d.

Theorem I′ follows immediately from Theorem II. The unsettled cases $n=4$, 5 corresponding to the unsolved Poincaré conjecture for the dimensions 3, 4.

In the cases $n=5$, 3 Theorem I follows from the same argument and the fact that $S^5-F=M-\partial M=S^1 \times (X-\partial X)$ is of the same homotopy type as S^1; then apply a result of J. Stallings [**9**].

Remark. The case $n=4$ is the only unsettled case but it is implied by the proof that $\pi_1(S^4-F)=Z$. This shows that an example with similar properties of the example of Mazur [**5**] is impossible.

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