ON CARDINALITIES OF ULTRAPRODUCTS

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Introduction. In the theory of models, the ultraproduct (or prime reduced product) construction has been a very useful method of forming models with given properties (see, for instance, [2]). It is natural to ask what the cardinality of an ultraproduct is when we are given the cardinalities of the factors. In this paper we obtain some new results in that direction; however, the questions stated explicitly in [2, p. 208], are still open.

Let us first mention briefly some of the known results. Throughout this note we shall let D be a nonprincipal ultrafilter over a set I of infinite power λ. Additional notation is explained in §1 below.

1. $\alpha \leqq \alpha^I/D \leqq \alpha^\lambda$ [2, p. 205].

2. If D is not countably complete, then $\prod_{i\in I} \alpha_i/D$ is either finite or of power at least 2^ω [2, p. 208].

3. If D is uniform, then $\lambda^I/D > \lambda$; moreover, $(2^{(\lambda)})^I/D = 2^\lambda$, where $2^{(\lambda)} = \sum_{\beta<\lambda} 2^\beta$ [2, p. 206].

4. There exists a D such that if α is infinite, then $\alpha^I/D = \alpha^\lambda$ [2, p. 207], [1, p. 399], and [3, p. 838]. (Two more general versions for products of cardinals are given in [1].)

We shall prove the following results.

THEOREM A. (i) *If α is infinite and D is not countably complete, then*

$$\alpha^I/D = (\alpha^I/D)^\omega.$$

(ii) *For any α, γ, and D,*

$$(\alpha^\gamma)^I/D \geqq (\alpha^I/D)^\gamma.$$

(iii) *If D is uniform then*

$$(\alpha^{(\lambda)})^I/D = (\alpha^I/D)^\lambda = \alpha^\lambda$$

where $\alpha^{(\lambda)} = \sum_{\beta<\lambda} \alpha^\beta$.

We introduce the notion of a (β, γ)-regular ultrafilter in §1, and use it to prove Theorem A and some more general results in §2.

1. **Regular ultrafilters.** We shall adopt all of the set-theoretical notation introduced in [1], including the notions of an ultraproduct $\prod_{i\in I} \alpha_i/D$ and ultrapower α^I/D of the cardinals α_i, α. We denote the set of all functions on X into Y by ${}^XY$. We let $S(X)$ be the set of all

subsets of X, and $S_\beta(\gamma)$ the set of all subsets of γ of power less than β. Assume hereafter that β, γ are infinite. If f is a function on X into $S(Y)$, then we define the function f^* on Y into $S(X)$ by

$$f^*(y) = \{x \in X : y \in f(x)\}, \qquad \text{for } y \in Y.$$

Thus we always have $f^{**} = f$.

DEFINITION 1.1. Let f be a function on I into $S(\gamma)$. We shall say that *f makes D (β, γ)-regular* if $f(i) \in S_\beta(\gamma)$ for all $i \in I$ and $f^*(\eta) \in D$ for all $\eta < \gamma$. D is *(β, γ)-regular* if there exists an f which makes D (β, γ)-regular.

LEMMA 1.2. *Let g be a function on γ into $S(I)$. Then g^* makes D (β, γ)-regular if and only if $g(\eta) \in D$ for all $\eta < \gamma$ and $\bigcap_{\eta \in Y} g(\eta) = 0$ for all $Y \subseteq \gamma$ of power β.*

D is said to be *uniform* if every member of D is of power λ (cf. [2]). As pointed out in [2], the uniform ultrafilters are the only interesting ones as far as the problems considered here are concerned.

LEMMA 1.3. (i) *If D is not countably complete, then D is (ω, ω)-regular.*

(ii) *If $\beta > \gamma$, then D is (β, γ)-regular.*

(iii) *If D is uniform, then D is* $(\mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$*-regular.*

(iv) *If D is* $(\mathrm{cf}(\gamma), \mathrm{cf}(\gamma))$*-regular, then D is (γ, γ)-regular.*

REMARK. The notion of regularity has other simple properties which we shall not need here. For instance, if D is (β, γ)-regular and $\beta \leqq \beta'$, $\gamma \geqq \gamma'$, then D is (β', γ')-regular. If $\lambda, \beta < \gamma$, then D is not (β, γ)-regular. Moreover, if $\lambda < \mathrm{cf}(\gamma)$, then D is not (γ, γ)-regular.

The proofs of Lemmas 1.2 and 1.3 above may easily be supplied by the reader.

2. **Cardinality theorems.**

THEOREM 2.1. *Suppose that f makes D (β, γ)-regular, and let β_i be the power of $f(i)$ for each $i \in I$. Then for any cardinals α_i, $i \in I$, we have*

$$\prod_{i \in I} (\alpha_i^{\beta_i})/D \geqq \left(\prod_{i \in I} \alpha_i/D \right)^\gamma.$$

PROOF. For each i, let g_i be a one-one function on ${}^{f(i)}\alpha_i$ into $\alpha_i^{\beta_i}$. Define the function g on ${}^\gamma(P_{i \in I}\, \alpha_i)$ into $P_{i \in I}\, {}^{f(i)}\alpha_i$ by

$$g(\langle h_\eta \rangle_{\eta < \gamma}) = k,$$

where

$$k(i) = g_i(\langle h_\eta(i) \rangle_{\eta \in f(i)}) \qquad \text{for each } i \in I.$$

Now consider two arbitrary elements $h=\langle h_\eta\rangle_{\eta<\gamma}$, $h'=\langle h'_\eta\rangle_{\eta<\gamma}$ of ${}^\gamma(P_{i\in I}\,\alpha_i)$, and let $g(h)=k$, $g(h')=k'$. Suppose that there exists $\eta<\gamma$ such that $h_\eta\not\equiv_D h'_\eta$. Whenever $h_\eta(i)\neq h'_\eta(i)$ and $\eta\in f(i)$ we have $k(i)\neq k'(i)$. Since $f^*(\eta)\in D$ it follows that $k\not\equiv_D k'$. The desired inequality follows.

Theorem A follows from Lemma 1.3 and Theorem 2.1. Indeed we have the following more general result.

THEOREM 2.2. (i) *If each cardinal α_i is infinite and D is not countably complete, then*

$$\prod_{i\in I}\alpha_i/D=\left(\prod_{i\in I}\alpha_i/D\right)^\omega.$$

(ii) *For any α_i, γ, and D, we have*

$$\prod_{i\in I}(\alpha_i^\gamma)/D\geqq\left(\prod_{i\in I}\alpha_i/D\right)^\gamma.$$

(iii) *If D is uniform and* $\mathrm{cf}(\gamma)=\mathrm{cf}(\lambda)$, *then*

$$\prod_{i\in I}(\alpha_i^{(\gamma)})/D\geqq\left(\prod_{i\in I}\alpha_i/D\right)^\gamma.$$

(iv) *If D is uniform and, for each $i\in I$, $\{j\in I:\alpha_j\geqq\alpha_i\}\in D$, then*

$$\prod_{i\in I}(\alpha_i^{(\lambda)})/D=\left(\prod_{i\in I}\alpha_i/D\right)^\lambda=\left(\prod_{i\in I}\alpha_i\right)^\lambda.$$

PROOF. We observe that under the hypotheses of Theorem 2.1,

$$\prod_{i\in I}(\alpha_i^{(\beta_i)})/D\geqq\left(\prod_{i\in I}\alpha_i/D\right)^\gamma.$$

Then (i), (ii), and (iii) follow using Lemma 1.3 with $\beta=\gamma=\omega$, $\beta=\gamma^+$, and $\beta=\gamma$, respectively. To prove (iv), we note that $\alpha_i\leqq\prod_{j\in I}\alpha_j/D$ for each $i\in I$, and hence

$$\left(\prod_{i\in I}\alpha_i/D\right)^\lambda\leqq\prod_{i\in I}(\alpha_i^{(\lambda)})/D\leqq\prod_{i\in I}(\alpha_i^{(\lambda)})\leqq\left(\prod_{i\in I}\alpha_i\right)^\lambda$$

$$\leqq\left(\prod_{i\in I}\left(\prod_{j\in I}\alpha_j/D\right)\right)^\lambda=\left(\prod_{i\in I}\alpha_i/D\right)^\lambda.$$

THEOREM 2.3. *Suppose that each n_i is a finite cardinal and D is not* *countably complete. Let* $q \in {}^{\omega}\omega$ *be such that* $\lim_{m\to\infty} q(m) = \infty$. *If* $\prod_{i\in I} n_i/D$ *is infinite, then*

$$\prod_{i\in I} (n_i^{q(n_i)})/D \geqq \left(\prod_{i\in I} n_i/D \right)^{\omega}.$$

PROOF. Since $\prod_{i\in I} n_i/D$ is infinite, we have $\{i\in I: n_i > m\} \in D$ for each $m<\omega$. Let f be the function on I into $S_\omega(\omega)$ defined by

$$f(i) = \{0, 1, \cdots, q(n_i) - 1\}, \qquad \text{for each } i \in I.$$

For each $r<\omega$, there is a greatest $m<\omega$ such that $q(m) \leqq r$, and hence

$$f^*(r) = \{i \in I: q(n_i) > r\} \supseteq \{i \in I: n_i > m\} \in D.$$

Thus f makes D (ω, ω)-regular. The result now follows from Theorem 2.1 with $\beta=\gamma=\omega$.

Notice that the result 2 stated in the introduction follows from the above theorem, because if $\prod_{i\in I} \alpha_i/D$ is infinite then we may choose n_i such that $n_i^{n_i} \leqq \alpha_i$ and $\prod_{i\in I} n_i/D$ is infinite.

We conclude with some historical remarks. The (ω, λ)-regular ultrafilters have been considered in the literature, for instance in [1], [2], [3], [6]. The result 4 stated in the introduction was shown in [1], [2], [3] to hold for all (ω, λ)-regular ultrafilters D. It is not difficult to show that any ultrafilter D which belongs to the class $Q(\alpha^+)$ defined in [4] is (ω, α)-regular. However, by Theorem 5.1 of [5], there is an (ω, λ)-regular D which is not a member of $Q(\omega_2)$.

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