REFERENCES

1. S. Hellerstein, *On the distribution of values of a meromorphic function and a theorem of H. S. Wilf* (to appear).
2. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Press, London, 1939.
3. H. S. Wilf, *The argument of an entire function*, Bull. Amer. Math. Soc. 67 (1961), 488–489.

UNIVERSITY OF WISCONSIN

DECOMPOSITION OF RIEMANNIAN MANIFOLDS[1]

BY H. WU

Communicated by I. Singer, February 24, 1964

A decomposition problem in geometry is the following: given a C^∞ manifold M on which is defined an affine connection without torsion, under what conditions in terms of the holonomy group will M be affinely diffeomorphic to a direct product of two other affinely connected manifolds? Here, we give a complete solution to the problem in the case the connection is one induced by a (definite or indefinite) riemannian metric—Main Theorem.

For a more detailed discussion of this general problem, see [1] and [2], especially §5.1–5.3 of [1], §I of [2], as well as Theorem 1 below. Clarifications of various concepts introduced in this paper are also given therein.

We need some definitions. Inner products on vector spaces and riemannian metrics on manifolds can be either definite or indefinite in this paper. A subspace V' of an inner-product space V is said to be nondegenerate (resp., degenerate, isotropic) iff the restriction of the inner product to V' is nondegenerate (resp., degenerate, zero). The action of a connected Lie group G acting on V will be said to be nondegenerately reducible iff G leaves invariant a proper nondegenerate subspace of V. The maximal subspace of V on which G acts as the identity is called the maximal trivial space of G in V. From now on, we fix a point $m \in M$. Then the holonomy group of M will always be understood to be acting on M_m (tangent space to M at m), so that in this case all references to M_m will be omitted. Finally, a pair $(\phi, M^1 \times M^2)$ is called an affine decomposition (resp. an isometric decomposition) of the affinely connected manifold M (resp. rieman-

[1] Research supported by the National Science Foundation.

nian manifold M) iff M^1, M^2 are affinely connected manifolds (resp. riemannian manifolds) and $\phi: M \to M^1 \times M^2$ is an affine diffeo, i.e. a diffeo that preserves the affine connections (resp. an isometry).

MAIN THEOREM. *Let M be a simply connected complete riemannian manifold. Then the following are equivalent*:

(a) *M admits an affine decomposition.*

(b) *M admits an isometric decomposition.*

(c) *The holonomy group of M is nondegenerately reducible.*

In reality, this theorem is only a rough statement of two very precise results. (c)$\Rightarrow$(b) is a consequence of the deRham Decomposition Theorem proved in [1] which gives the isometry explicitly. (a)$\Rightarrow$(c) is a consequence of Theorem 2 below which determines exactly the structure of an arbitrary affine decomposition. (b)$\Rightarrow$(a) is of course trivial. We therefore proceed to state and prove Theorem 2.

From now on, we agree to call a subspace N of M_m which is left invariant by the holonomy group of M simply an invariant subspace. Given such an N, we denote by $\mu(N)$ the unique integral manifold through m of the distribution on M obtained by parallel translating N over M. We shall need the following theorem which is a restatement of (*) of §5.2 of [1] in the form convenient for our immediate purpose. Its proof consists of nothing more than a formal rephrasing here and there of the proof of the deRham Theorem given in [1]. For further details, see §5.1–5.3 of [1].

THEOREM 1. *Let M be a simply connected manifold with a complete torsionless connection. Fix a point $m \in M$ and consider all curves $\gamma: [0, 1] \to M$ with $\gamma(0) = m$. If R is the curvature tensor of M, let $(\gamma R): M_m \wedge M_m \to \mathrm{Hom}(M_m, M_m)$ be the linear transformation defined by*:

$$(\gamma R)_{xy} = \gamma^{-1} \cdot R_{\gamma(x),\gamma(y)} \cdot \gamma,$$

where $x, y \in M_m$ and γ stands for the isomorphism $M_m \to M_{\gamma(1)}$ induced by parallel translation along γ. Suppose V and W are invariant subspaces of M_m such that $M_m = V \oplus W$, then M is affinely diffeomorphic to $\mu(V) \times \mu(W)$ iff $(\gamma R)_{vw} \equiv 0$ for all $v \in V$, $w \in W$ and for all such γ.

THEOREM 2. *Let M be a simply connected complete riemannian manifold, and let $(\phi, M^1 \times M^2)$ be an affine decomposition of M. Fix $m \in M$ and let $\phi(m) = (m_1, m_2)$, $(d\phi)^{-1}(M^1_{m_1}) = P$, $(d\phi)^{-1}(M^2_{m_2}) = Q$. Then there exist four invariant subspaces V, W, E, F of M_m with these properties*:

(a) *$P = V \oplus E$, $Q = W \oplus F$.*

(b) *V, W, $U\equiv V\oplus W$, $D\equiv E\oplus F$ are all nondegenerate.*

(c) *The maximal trivial space of the holonomy group of M in each of U, V, W, (resp., D, E, F) is isotropic (resp., D, E, F itself).*

(d) *If $U^\perp$ (resp. $V^\perp$) is the orthogonal complement of U in M_m (resp. V in U) and if α (resp. β) is the maximal trivial space of the holonomy group of M in U (resp. in V), then $D\subseteq U^\perp\oplus\alpha$ (resp. $W\subseteq V^\perp\oplus\beta$).*

In particular, ϕ induces an affine diffeo of M, M^1, M^2, with resp. $\mu(U)\times\mu(D)$, $\mu(V)\times\mu(E)$, $\mu(W)\times\mu(F)$; $\mu(U)$, $\mu(V)$, $\mu(W)$ are riemannian manifolds whose holonomy groups all have isotropic maximal trivial spaces, $\mu(E)$, $\mu(F)$ are flat and $\mu(D)$ is naturally an inner product space.

By virtue of Theorem 1, the proof of Theorem 2 is reduced to a problem in linear algebra. In order to eliminate the large number of quantifiers in the proof, we shall set up this *convention*: x, y, z will be generic symbols of elements of M_m, u, p, e, etc. will be generic symbols of elements of subspaces of M_m designated by the corresponding capital letters, i.e. U, P, E, etc., and R will be the generic symbol of the set of linear maps (γR) in Theorem 1. Note that Ambrose-Singer's theorem states the holonomy algebra of M as exactly the span of all $(\gamma R)_{xy}$, for all γ, all x, all y. Thus the holonomy group acts trivially on $V\subseteq M_m$ iff, in our abbreviated language, $R_{xy}V=0$, and V is invariant iff $R_{xy}V\subseteq V$. We may therefore restate the last part of Theorem 1 as: If $R_{xy}V\subseteq V$, $R_{xy}W\subseteq W$, then M is affinely diffeomorphic to $\mu(V)\times\mu(W)$ iff $R_{vw}\equiv 0$.

Proof of Theorem 2. The essential difficulty lies in proving (b), which is accomplished in (7) and (13) below. Notation as in the theorem, we first note:

$$(1)\qquad M_m = P\oplus Q,\qquad R_{xy}P\subseteq P,\qquad R_{xy}Q\subseteq Q,\qquad R_{pq}\equiv 0.$$

Let E' be the null subspace of P, i.e. $E'=\{e'\in P\colon \langle e', P\rangle=0\}$. Similarly let F' be the null subspace of Q. We claim:

$$(2)\qquad R_{xy}E' = R_{xy}F' = \{0\}.$$

For, let $e'\in E'$, we show $\langle R_{xy}e', z\rangle=0$. If $z\in P$, then $R_{xy}z\in P$, $\Rightarrow\langle R_{xy}e', z\rangle=-\langle e', R_{xy}z\rangle=0$ by definition of E'. If $z\in Q$, $\langle R_{xy}e', z\rangle=\langle R_{e'z}x, y\rangle=0$ by virtue of (1). This proves $R_{xy}E'=0$; similarly $R_{xy}F'=0$, proving (2).

Next, let $Q^\perp$, $P^\perp$ be the orthogonal complements of Q, P in M_m resp. Since $R_{pq}\equiv 0$, $\langle R_{xy}p, q\rangle=\langle R_{pq}x, y\rangle=0$. Hence:

$$(3)\qquad R_{xy}P\subseteq Q^\perp\cap P,\qquad R_{xy}Q\subseteq P^\perp\cap Q.$$

Now $E'\cap(Q^\perp\cap P)=\{0\}$ because $e'\in(E'\cap Q^\perp\cap P)\Rightarrow\langle e', P\rangle=0$ by definition of $E^\perp$, and $\langle e', Q\rangle=0$ since $e'\in Q^\perp$. So $\langle e', M_m\rangle=0$, $\Rightarrow e'=0$. Thus $Q^\perp\cap P$ is contained in a subspace V' of P such that $V'\oplus E'=P$. We claim V' is nondegenerate and invariant. The former is by definition of E'. To prove the latter, observe that $(3)\Rightarrow R_{xy}V'\subseteq R_{xy}P\subseteq Q^\perp\cap P\subseteq V'$. Applying the same reasoning to Q, we get:

$$(4) \qquad P = V' \oplus E', \qquad Q = W' \oplus F',$$

where V', W' are nondegenerate and invariant and $R_{xy}E'=R_{xy}F'=0$.

Now let E'' (resp. F'') be a maximal nondegenerate subspace of V' (resp. W') such that $R_{xy}E''=0$ (resp. $R_{xy}F''=0$). Let V (resp. W) be the orthogonal complement of E'' in V' (resp. of F'' in W'). We define:

$$E = E'' \oplus E', \qquad F = F'' \oplus F'.$$

Thus we may translate (4) into:

$$(5) \qquad P = V \oplus E, \qquad Q = W \oplus F,$$

where V, W are nondegenerate invariant subspaces of M_m, and the maximal trivial space of the holonomy group in each is isotropic; $R_{xy}E=R_{xy}F=0$.

At this point, let us note that if both $V=W=\{0\}$, M is flat and hence an inner-product space. We have nothing to prove in this case. So assume $V\neq\{0\}$. Also observe that: $R_{xy}E=R_{xy}F=0$ together with $\langle R_{xy}e, z\rangle=\langle R_{ez}x, y\rangle$ imply:

$$(6) \qquad R_{xe} = R_{yf} \equiv 0 \qquad \text{if } e \in E \text{ and } f \in F.$$

Define now: $U=V\oplus W$, $D=E\oplus F$. We claim:

$$(7) \qquad U \text{ is nondegenerate.}$$

Proof of (7). If $W=\{0\}$, we are done. So $W\neq\{0\}$. We shall assume U degenerate and deduce a contradiction. Let $0\neq u\in U$ such that $\langle u, U\rangle=0$; in particular $\langle u, u\rangle=0$. If $V^\perp$ is the orthogonal complement of V in M_m, then $u\in V^\perp$. Let $\{h_1, \cdots, h_l\}$ be an orthonormal basis of $V^\perp$ and may assume $u=h_1+h_2$, i.e. $\langle h_1, h_2\rangle=0$, $\langle h_1, h_1\rangle=-\langle h_2, h_2\rangle=1$. Let $u=u_v+u_w$ with respect to the decomposition $(V\oplus W)$ of U, and assert:

$$(8) \qquad R_{xy}u = 0.$$

For, if $d\in D$, $(5)\Rightarrow\langle R_{xy}u, d\rangle=-\langle u, R_{xy}d\rangle=0$, and if $s\in U$, then $\langle R_{xy}u, s\rangle=-\langle u, R_{xy}s\rangle=0$ since $R_{xy}s\in U$. So $\langle R_{xy}u, M_m\rangle=0$, proving (8).

Thus $R_{xy}u_v+R_{xy}u_w=0$. But $R_{xy}u_v\in V$ and $R_{xy}u_w\in W$; as $U=V\oplus W$ is a direct sum, this is possible iff $R_{xy}u_v=R_{xy}u_w=0$. Again by (5), the maximal trivial spaces of the holonomy group in V and W are isotropic. Hence:

$$\langle u_v, u_v\rangle = \langle u_w, u_w\rangle = 0. \tag{9}$$

We now assume orthonormal basis $\{v_1, \cdots, v_r\}$ and $\{w_1, \cdots, w_s\}$ of V and W have been so chosen that $u_v=v_1+v_2$, $u_w=w_1+w_2$ where, of course, $\langle v_1, v_2\rangle=0$, $\langle v_1, v_1\rangle=-\langle v_2, v_2\rangle=1$, $\langle w_1, w_2\rangle=0$, $\langle w_1, w_1\rangle=-\langle w_2, w_2\rangle=1$. (This is permissible since V, W are nondegenerate.) Hence, $w_1+w_2=(h_1+h_2)-(v_1+v_2)$. Since $\{v_1, \cdots, v_r, h_1, \cdots, h_l\}$ form an orthonormal basis of M_m, let

$$w_1 = \sum_i a_iv_i + \sum_j b_jh_j,$$
$$w_2 = \sum_i m_iv_i + \sum_j n_jh_j.$$

Since $0=\langle u, w_1\rangle=\langle u, w_2\rangle$ and $u=h_1+h_2$ by definition, we have

$$b_1 = b_2, \qquad n_1 = n_2. \tag{10}$$

Now from (1) and (5), $R_{w_1v_k}\equiv 0$ for all k. Hence $\langle R_{xy}w_1, v_k\rangle=0$, $\Rightarrow\langle R_{xy}(\sum_i a_iv_i), v_k\rangle=0$ for all k. Since $R_{xy}V\subseteq V$, this implies $R_{xy}(\sum_i a_iv_i)=0$. By (5) again, $(\sum_i a_iv_i)$ is then an isotropic vector. Similarly, $(\sum_i m_iv_i)$ is isotropic; so we have

$$\left\langle \sum_i a_iv_i, \sum_i a_iv_i\right\rangle = \left\langle \sum_i m_iv_i, \sum_i m_iv_i\right\rangle = 0. \tag{11}$$

On the other hand, by equating coefficients in $w_1+w_2=(h_1+h_2)-(v_1+v_2)$, we get:

$$\begin{aligned} a_1 + m_1 &= a_2 + m_2 = -1, \\ b_1 + n_1 &= b_2 + n_2 = 1, \\ a_3 + m_3 &= \cdots = a_r + m_r = 0, \\ b_3 + n_3 &= \cdots = b_s + n_s = 0. \end{aligned} \tag{12}$$

Using (11) and the first two equations of (12), it is simple to show that $a_1=a_2$, $m_1=m_2$. Combining this with (10) and the last two equations of (12), we can write:

$$w_1 = a(v_1+v_2) + \sum_{i\geqq 3} a_iv_i + b(h_1+h_2) + \sum_{j\geqq 3} b_jh_j,$$
$$w_2 = -(a+1)(v_1+v_2) - \sum_{i\geqq 3} a_iv_i - (b-1)(h_1+h_2) - \sum_{j\geqq 3} b_jh_j.$$

These expressions for w_1, w_2 together with (11) entail the fact that $\langle \sum_{i\geqq 3} a_i v_i, \sum_{i\geqq 3} a_i v_i\rangle = 0$. Consequently, $1 = \langle w_1, w_1\rangle = \langle \sum_{j\geqq 3} b_j h_j, \sum_{j\geqq 3} b_j h_j\rangle$. But by choice $\langle w_1, w_2\rangle = 0$, or equivalently, $\langle \sum_{j\geqq 3} b_j h_j, -\sum_{j\geqq 3} b_j h_j\rangle = 0$. This is a contradiction and (7) is proved.

(13) D is nondegenerate.

PROOF OF (13). We have $M_m = U \oplus D$, $U \neq \{0\}$. By (7), U is nondegenerate; let $U^\perp$ be the orthogonal complement of U in M_m and $M_m = U \oplus U^\perp$.

(14) $$R_{xy}D = R_{xy}U = 0.$$

The first part is contained in (5). For the second part, if $u^\perp \in U^\perp$, let $u^\perp = u + u_d$ relative to $M_m = U \oplus D$. So $R_{xy}u^\perp = R_{xy}u + R_{xy}u_d = R_{xy}u \in U$. But $R_{xy}u \in U$ since U is invariant. Thus $U \cap U^\perp = \{0\} \Rightarrow R_{xy}u^\perp = 0$, proving (14).

Now we assume (13) false and let $0 \neq d^* \in D$ such that $\langle d^*, D\rangle = 0$; in particular $\langle d^*, d^*\rangle = 0$. Let $d^* = d + d^\perp$ relative to the decomposition $M_m = U \oplus U^\perp$. By (14), $0 = R_{xy}d^* = R_{xy}d + R_{xy}d^\perp = R_{xy}d$ and hence (5) implies $\langle d, d\rangle = 0$. Consequently, $\langle d^\perp, d^\perp\rangle = 0$. So let $\{u_1, \cdots, u_r\}$ and $\{u_1^\perp, \cdots, u_s^\perp\}$ be orthonormal basis of U and $U^\perp$ such that $d = u_1 + u_2$, $d^\perp = u_1^\perp + u_2^\perp$, i.e. $\langle u_1, u_2\rangle = 0$, $\langle u_1, u_1\rangle = -\langle u_2, u_2\rangle = 1$, $\langle u_1^\perp, u_2^\perp\rangle = 0$, $\langle u_1^\perp, u_1^\perp\rangle = -\langle u_2^\perp, u_2^\perp\rangle = 1$. Now if $\delta' \in D$, let $\delta' = \delta + \delta^\perp$ relative to $U \oplus U^\perp$. As before, $R_{xy}\delta' = 0$ and $R_{xy}\delta^\perp = 0 \Rightarrow R_{xy}\delta = 0$, $\Rightarrow \langle \delta, \delta\rangle = 0$. If $\delta = \sum_i a_i u_i$, then $a_1^2 - a_2^2 + \langle \sum_{i\geqq 3} a_i u_i, \sum_{i\geqq 3} a_i u_i\rangle = 0$. Since also $(\delta' - d^*) \in D$ and $\delta' - d^* = (\delta - d) + (\delta^\perp - d^\perp)$ is the decomposition of $(\delta' - d^*)$ relative to $M_m = U \oplus U^\perp$, again $\langle \delta - d, \delta - d\rangle = 0$, i.e. $(a_1 - 1)^2 - (a_2 - 1)^2 + \langle \sum_{i\geqq 3} a_i u_i, \sum_{i\geqq 3} a_i u_i\rangle = 0$. It is then clear that $a_1 = a_2$ so that $\delta = a(u_1 + u_2) + \sum_{i\geqq 3} a_i u_i$. Since $\langle d^*, D\rangle = 0$, we have $\langle d^*, \delta'\rangle = 0$. This implies $\langle \delta, d\rangle + \langle \delta^\perp, d^\perp\rangle = 0$. By the special form of δ, $\langle \delta^\perp, d^\perp\rangle = 0$. Recalling that $d^\perp = u_1^\perp + u_2^\perp$, $\delta^\perp = b(u_1^\perp + u_2^\perp) + \sum_{l\geqq 3} b_l u_l^\perp$. We therefore have:

(15) $$\delta' \in D \Rightarrow \delta' = a(u_1 + u_2) + \sum_{i\geqq 3} a_i u_i + b(u_1^\perp + u_2^\perp) + \sum_{l\geqq 3} b_l^\perp u_l^\perp.$$

Now let $\{d_1 = d^*, d_2, \cdots, d_s\}$ be a basis of D. By virtue of (15), we may assume (after subtracting a suitable multiple of d^* from each of $d_2, \cdots, d_s$) that each of $d_2, \cdots, d_s$ does not involve $(u_1^\perp + u_2^\perp)$ in its expansion relative to $\{u_1, \cdots, u_r, u_1^\perp, \cdots, u_s^\perp\}$. Hence:

$$\operatorname{span}\{d_2, \cdots, d_s\} \subseteq \operatorname{span}\{u_1 + u_2, u_3, \cdots, u_r, u_3^\perp, \cdots, u_s^\perp\}.$$

The dimension of the left side is $(s-1)$, that of the right side is $(r+s)-3$. Also, the dimension of $\operatorname{span}\{u_1 + u_2, u_3, \cdots, u_r\}$ is $(r-1)$. Therefore, because of purely dimensional reasons,

$$\operatorname{span}\{d_2, \cdots, d_s\} \cap \operatorname{span}\{u_1 + u_2, u_3, \cdots, u_r\} \neq \{0\}.$$

But the first factor is a subspace of D and the second factor is a subspace of U, and this contradicts $U \cap D = \{0\}$. Hence D is nondegenerate and (13) is proved.

Finally, as a matter of formality, we round off the proof of Theorem 2. (a) and (c) follow from (5). (b) is immediate from (5), (7) and (13). The assertion that $w \subseteq (V^\perp \oplus \beta)$ of (d) is contained in (9), and the assertion about $D \subseteq (U^\perp \oplus \alpha)$ can be proved similarly. The last assertion of the theorem is a consequence of (1), (6), (a)–(c) and Theorem 1. Q.E.D.

COROLLARY 1. *If the maximal trivial space of the holonomy group of M is isotropic and $(\phi, M^1 \times M^2)$ is an affine decomposition of M, then the connections of M^1 and M^2 are riemannian connections.*

COROLLARY 2. *If the maximal trivial space of the holonomy group of M is zero, then every affine decomposition of M is an isometric decomposition.*

For completeness, we state here the localized versions of the Main Theorem.

THEOREM 3. *Let M be a riemannian manifold; then the following are equivalent:*

(a) *Locally M admits an affine decomposition.*

(b) *Locally M admits an isometric decomposition.*

(c) *The identity component of the holonomy group of M is nondegenerately reducible.*

EXAMPLE. We now show that Theorem 2 is the best one can say about a general affine decomposition. It is known that [2, Theorem 3] $\boldsymbol{R}^4$ can be given the structure of a globally symmetric riemannian manifold with holonomy group

$$\begin{bmatrix} 1 & -t & 0 & t \\ t & 1 & -t & 0 \\ 0 & -t & 1 & t \\ t & 0 & -t & 1 \end{bmatrix}, \qquad t \in \boldsymbol{R},$$

where it is understood that the canonical basis $(f_1, \cdots, f_4)$ of $\boldsymbol{R}^4_o$ (tangent space to $\boldsymbol{R}^4$ at the origin) has signature $(+, +, -, -)$. Note that the maximal trivial space is $\operatorname{span}\{f_1+f_3, f_2+f_4\}$.

Now consider $M = \boldsymbol{R}^4 \times \boldsymbol{R}^4 \times \boldsymbol{R}^2$, where the first two factors are each given the above exotic riemannian structure and $\boldsymbol{R}^2$ is just an inner-product space of signature $(+, -)$. Let us agree that if $\{e_1, \cdots, e_{10}\}$ is the natural basis of $M_o = \boldsymbol{R}_o^4 \oplus \boldsymbol{R}_o^4 \oplus \boldsymbol{R}_o^2$, the signature of the metric is $(+, +, -, -, +, +, -, -, +, -)$. Let

$$\begin{aligned} V &= \operatorname{span}\{e_1, e_2, e_3, e_4\}, \\ W &= \operatorname{span}\{e_5 + (e_1 + e_3), e_6, e_7 - (e_1 + e_3), e_8\} \\ E &= \operatorname{span}\{e_2 + e_4 + e_9\}, \\ F &= \operatorname{span}\{e_6 + e_8 + e_9 + e_{10}\}. \end{aligned}$$

Furthermore, define $U = V \oplus W$, $D = E \oplus F$, $M' = \mu(V \oplus E)$, $M^2 = \mu(W \oplus F)$, then clearly every assertion of Theorem 2 is satisfied for these $D, \cdots, W$.

The pathological features of this affine decomposition $M^1 \times M^2$ of M are the following: (1) $\langle F, F\rangle = 0$, but (2) $\langle E, F\rangle \neq 0$, (3) $\langle V \oplus E, W \oplus F\rangle \neq 0$, (4) $\langle V, W\rangle \neq 0$, (5) $\langle U, D\rangle \neq 0$, (6) $\langle V, E\rangle \neq 0$, and (7) $\langle W, F\rangle \neq 0$.

Bibliography

1. H. Wu, *On the deRham Decomposition Theorem*, Illinois J. Math. (to appear).
2. ———, *Some remarks on holonomy*, Illinois J. Math. (to appear).

Massachusetts Institute of Technology