THE REAL VALUES OF AN ENTIRE FUNCTION[1]

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Communicated by W. Rudin, March 30, 1964

The object of this note is to prove the following

THEOREM. *Let $f(z)$ be a nonconstant entire function of order ρ, and let $\Phi(r)$ be the number of points on the circumference $|z|=r$ at which $f(z)$ is real. Then*

$$\limsup_{r\to\infty}\frac{\log\Phi(r)}{\log r}=\rho. \tag{1}$$

We will denote the left-hand side of (1) by κ. Before we proceed with the proof we remark that H. S. Wilf has shown in [3] that $\kappa\geqq\rho$. He then raised the question whether strict inequality could ever hold. The above result settles this problem.

Our proof consists of two parts. In the first, we again prove that $\kappa\geqq\rho$, but by a considerably simpler method than the one given in [3]. The present method has the advantage that it may be used to count not only the real values, but also the values which belong to an arbitrary given unbounded curve. This latter generalization, along with a similar investigation for meromorphic functions, will be published elsewhere [1]. In the second part of the proof we show that $\kappa\leqq\rho$.

PROOF OF THE THEOREM.

Part (i): $\kappa\geqq\rho$. Put $w=f(z)$ and denote by $n(r, w_0)$ the number of w_0 points (multiplicity included) of $f(z)$ in the disc $|z|\leqq r$. A well-known theorem of Borel asserts that for all but at most one finite value w_0,

$$\limsup_{r\to\infty}\frac{\log n(r,w_0)}{\log r}=\rho. \tag{2}$$

Let w_0 be any real value for which (2) holds. Then, an elementary argument implies that there exists an unbounded, increasing sequence $\{r_k\}$ such that

$$\lim_{k\to\infty}\frac{\log n(r_k,w_0)}{\log r_k}=\rho, \tag{3}$$

while $f(z)\neq w_0$ on the circumferences $|z|=r_k$ $(k=1,2,\cdots)$.

[1] Work supported by NSF grant GP-2159.

By the argument principle, for each $k=1, 2, \cdots$, the winding number of the curve $w=f(r_k e^{i\theta})$, $0\leqq\theta\leqq 2\pi$, about the point w_0 in the w-plane is given by $n(r_k, w_0)$. Since w_0 is real, this curve must intersect the real axis at least $2n(r_k, w_0)$ times as θ varies from 0 to 2π. Hence

$$\Phi(r_k) \geqq 2n(r_k, w_0), \tag{4}$$

and in view of (3) it follows that ρ cannot exceed κ.

Part (ii): $\kappa\leqq\rho$. Let

$$f(z) = \sum a_n z^n. \tag{5}$$

Since f is of order ρ, so is [2, p. 253],

$$f^*(z) = \sum \mid a_n \mid z^n. \tag{6}$$

We set $a_n=\alpha_n+i\beta_n$ (α_n, β_n real, $n=1, 2, \cdots$), and introduce

$$g_r(\theta) = \operatorname{Im} f(re^{i\theta}) = \sum(\alpha_n \sin n\theta + \beta_n \cos n\theta) r^n. \tag{7}$$

For fixed $r>0$, $g_r(\theta)$ is an entire function of θ, since for $|\theta|\leqq\theta_0$,

$$\sum \mid \alpha_n \sin n\theta + \beta_n \cos n\theta \mid r^n \leqq \sum \mid a_n \mid (e^{\theta_0} r)^n = f^*(e^{\theta_0} r). \tag{8}$$

In particular for $|\theta|\leqq 2\pi$, we have from (7) and (8), and the fact that f^* is of order ρ,

$$\mid g_r(\theta) \mid \leqq M \exp\left\{(e^{2\pi} r)^{\rho+\epsilon}\right\}, \tag{9}$$

with $\epsilon>0$ arbitrary, $M=M(\epsilon)$, M independent of r.

Now let $n_r(t)$ denote the number of zeros of $g_r(\theta)$ in $|\theta|\leqq t$. It may be assumed that $g_r(0)\neq 0$. (Indeed, in the contrary case, we carry out an initial rotation which in general depends on r, but leaves $|a_n|$ unchanged.)

Then, by Jensen's formula,

$$\begin{aligned} \int_0^{2\pi} \frac{n_r(t)}{t} dt &= -\log \mid g_r(0) \mid + \frac{1}{2\pi} \int_0^{2\pi} \log \mid g_r(2\pi e^{i\phi}) \mid d\phi \\ &\leqq 2\left\{\log M + (e^{2\pi} r)^{\rho+\epsilon}\right\}. \end{aligned} \tag{10}$$

Hence,

$$n_r(\pi) \leqq \frac{2}{\log 2} \left\{\log M + (e^{2\pi} r)^{\rho+\epsilon}\right\}. \tag{11}$$

Since the number $n_r(\pi)$ includes, in particular, the real zeros of $g_r(\theta)$ with $|\theta|\leqq\pi$, it follows from (7) that $\Phi(r)\leqq n_r(\pi)$ and, in view of (11), that $\kappa\leqq\rho$.

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2. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Press, London, 1939.
3. H. S. Wilf, *The argument of an entire function*, Bull. Amer. Math. Soc. **67** (1961), 488–489.

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DECOMPOSITION OF RIEMANNIAN MANIFOLDS[1]

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Communicated by I. Singer, February 24, 1964

A decomposition problem in geometry is the following: given a C^∞ manifold M on which is defined an affine connection without torsion, under what conditions in terms of the holonomy group will M be affinely diffeomorphic to a direct product of two other affinely connected manifolds? Here, we give a complete solution to the problem in the case the connection is one induced by a (definite or indefinite) riemannian metric—Main Theorem.

For a more detailed discussion of this general problem, see [1] and [2], especially §5.1–5.3 of [1], §I of [2], as well as Theorem 1 below. Clarifications of various concepts introduced in this paper are also given therein.

We need some definitions. Inner products on vector spaces and riemannian metrics on manifolds can be either definite or indefinite in this paper. A subspace V' of an inner-product space V is said to be nondegenerate (resp., degenerate, isotropic) iff the restriction of the inner product to V' is nondegenerate (resp., degenerate, zero). The action of a connected Lie group G acting on V will be said to be nondegenerately reducible iff G leaves invariant a proper nondegenerate subspace of V. The maximal subspace of V on which G acts as the identity is called the maximal trivial space of G in V. From now on, we fix a point $m \in M$. Then the holonomy group of M will always be understood to be acting on M_m (tangent space to M at m), so that in this case all references to M_m will be omitted. Finally, a pair $(\phi, M^1 \times M^2)$ is called an affine decomposition (resp. an isometric decomposition) of the affinely connected manifold M (resp. rieman-

[1] Research supported by the National Science Foundation.