ON A STATIONARY APPROACH TO SCATTERING PROBLEM[1]

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1. Let H_p, $p=1, 2$, be self-adjoint operators in a Hilbert space $\mathfrak{H}$ satisfying the condition

$$(1) \qquad (H_1 - z)^{-1} - (H_0 - z)^{-1} \in \boldsymbol{T}(\mathfrak{H}), \qquad z \in \rho(H_0) \cap \rho(H_1).$$

Here, $\boldsymbol{T}(\mathfrak{H})$ denotes the trace class of completely continuous operators in $\mathfrak{H}$ and $\rho(H_p)$ the resolvent set of H_p. The perturbation theory of absolutely continuous (abbr. a.c.) parts of H_p as well as the theory of wave and scattering operators has recently been studied independently by de Branges [2], Birman and Kreĭn [1], and Kato [3]. In [1] and [3] the problem was considered from the viewpoint of the scattering theory. In particular, the wave operators $W_\pm$ were proved to exist and hence to be partially isometric operators which give the unitary equivalence of a.c. parts of H_0 and H_1. In [2], on the contrary, similar partially isometric operators $\hat{W}_\pm$ were constructed somewhat explicitly and *without* referring to the limit of wave operator type. The purpose of the present note is to study the latter approach from a viewpoint of the scattering theory and to see that the so-called time-independent or stationary approach to the theory of wave and scattering operators can be made possible under the condition (1). In a simpler case, a similar study was made in [4]. Our construction of the operator similar to $\hat{W}_\pm$, i.e. the operator given by the right side of (9), is similar to but slightly different from that given in [2]. In particular, the use of the auxiliary operator I in [2] is avoided. Furthermore, the construction of the operators π_0 and π_1 in 3 might be a little more explicit than that of the corresponding operators given in [2].

2. Let $\mathfrak{C}$ be a separable Hilbert space and let $\boldsymbol{T}_p \equiv \boldsymbol{T}_p(\mathfrak{C}) \subset \boldsymbol{T}(\mathfrak{C})$ be the set of all non-negative operators in $\boldsymbol{T}(\mathfrak{C})$. The trace norm will generally be denoted by $\tau(\ )$. Let μ be a $\boldsymbol{T}_p$-valued measure defined for bounded Borel sets of the reals R^1. Then the set function ρ, first defined at each bounded Borel set e as $\rho(e) = \tau(\mu(e))$ and then ex-

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tended by additivity to the Borel field of R^1, is a σ-finite (nonnegative) measure.

The $L^2(\mu)$ space over μ is defined as in [2]. In particular, the set $L_0^2(\mu)$ of all functions $f(x)$ from R^1 into $\mathfrak{C}$ such that $\int_{-\infty}^{\infty}|f(x)|^2 d\rho(x) < \infty$ forms a dense subset of $L^2(\mu)$ (after identification of functions equivalent in a certain sense). The space $L^2(\mu)$ can be identified with the direct sum of the L^2 spaces over the absolutely continuous and singular components of μ: $L^2(\mu) = L^2(\mu_{ac}) \oplus L^2(\mu_s)$. The space $L^2(\mu_{ac})$ is then the a.c. subspace of $L^2(\mu)$ with respect to the multiplication operator by x in $L^2(\mu)$.

In what follows we assume as in [2] that every $\boldsymbol{T}_p$-valued measure μ satisfies the condition

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, d\rho(x) < \infty. \tag{2}$$

For such μ, the $\boldsymbol{T}$-valued function $\phi_\mu(z)$ of a complex variable z, $\operatorname{Im} z \neq 0$, is defined as

$$\phi_\mu(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{xz+1}{(x-z)(1+x^2)}\, d\mu(x).$$

(The integral on the right may be regarded as the (improper) integral of a scalar function with respect to a vector-valued measure. Here, we note that the use of the coordinate representation in $\mathfrak{C}$ with respect to a complete orthonormal set allows us to make the definition of $L^2(\mu)$ space as well as the interpretation of all the integrals appearing in this note by means of the theory of integration of a vector-valued function with respect to a scalar measure.)

Now the following lemma, given in [2] and reformulated below in a slightly different form, will be our starting point.

LEMMA. (i) *The limits on reals of* $\phi_\mu(z)$:

$$\phi_\mu(x \pm i0) = \lim_{\epsilon \downarrow 0} \phi_\mu(x \pm i\epsilon), \qquad -\infty < x < \infty,$$

exist in the Schmidt norm in $\mathfrak{C}$ *almost everywhere with respect to the Lebesgue measure.*

(ii) *Let* μ *and* ν *be* $\boldsymbol{T}_p$*-valued measures both satisfying the condition such as* (2). *Let there exist self-adjont operators* α *and* β *in* $\mathfrak{C}$ *such that*

$$\{\alpha + \phi_\mu(z)\}\{\beta + \phi_\nu(z)\} = \{\beta + \phi_\nu(z)\}\{\alpha + \phi_\mu(z)\} = -1 \tag{3}$$

holds for every nonreal z and put

$$w(z) = \alpha + \phi_\mu(z), \qquad w_\pm(x) = w(x \pm i0).$$

Then, the mapping which assigns $w_\pm(x)f(x)$ *to each* $f(x)\in L_0^2(\mu_{ac})$ *is well-defined as an isometric mapping* $L_0^2(\mu_{ac})$ *onto* $L_0^2(\nu_{ac})$ *and hence can be extended uniquely to a partially isometric operator* $\Omega_\pm$ *from* $L^2(\mu)$ *into* $L^2(\nu)$ *with the initial set* $L^2(\mu_{ac})$ *and the final set* $L^2(\nu_{ac})$. *Therefore, if we denote the operators of the multiplication by* x *in* $L^2(\mu)$ *and* $L^2(\nu)$ *by* A *and* B, *respectively, then each of* $\Omega_\pm$ *gives the unitary equivalence between the a.c. parts of* A *and* B.

(iii) *Under the same assumption as in* (ii), *there exists a uniquely determined unitary operator* T *from* $L^2(\mu)$ *onto* $L^2(\nu)$ *such that* T *maps* $(x-z)^{-1}c\in L_0^2(\mu)$ *to* $w(z)(x-z)^{-1}c\in L_0^2(\nu)$ *for every nonreal* z *and* $c\in\mathfrak{C}$.

Now, the following theorem establishes a connection of $\Omega_\pm$ with the "asymptotic limit" of wave operator type.

THEOREM 1. *Let* μ, ν, A, B *and* T *be as in the lemma. Then, we have*

$$\Omega_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itB)T\exp(-itA)P,$$

where P *is the orthogonal projection in* $L^2(\mu)$ *onto its subspace* $L^2(\mu_{ac})$.

The proof of Theorem 1 is an adaptation of the arguments given in Kato [3, §5] which essentially prove Theorem 1 in the case of dim $\mathfrak{C}=1$. In particular, we get a kind of representation of T such as (4.5) of [3].

3. We shall next apply the foregoing consideration to the theory of wave operators. We shall begin with the following theorem which is deduced from Theorem 1 in a routine way.

THEOREM 2. *Let* H_p, $p=0, 1$, *be self-adjoint operators in a Hilbert space* $\mathfrak{H}$ *and* P_p *the orthogonal projection onto the a.c. subspace* $\mathfrak{M}_p$ *of* $\mathfrak{H}$ *with respect to* H_p. *Furthermore, let there exist a separable Hilbert space* $\mathfrak{C}$, $T_p(\mathfrak{C})$*-valued measure* μ *and* ν, *and unitary operators* π_0 *and* π_1 *from* $\mathfrak{H}$ *onto* $L^2(\mu)$ *and* $L^2(\nu)$, *respectively, such that* (*A and B are used as in Theorem* 1): (i) $H_0=\pi_0^{-1}A\pi_0$, $H_1=\pi_1^{-1}B\pi_1$; *and* (ii) μ *and* ν *satisfy the relation* (3) *with certain self-adjoint* α *and* β. *Then, the wave operator*

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \exp(itH)\exp(-itH_0)P_0$$

exists if and only if there exists a unitary operator $U_\pm$ *in* $\mathfrak{H}$ *such that*: (a) $H_1U_\pm=U_\pm H_1$; *and* (b) $\lim_{t\to\pm\infty}(\pi_1^{-1}T\pi_0-U_\pm)\exp(-itH_0)u=0$ *for each* $u\in\mathfrak{M}_0$. *In this case we have*

$$W_\pm = U_\pm^{-1}\pi_1^{-1}\Omega_\pm\pi_0$$

and hence $W_\pm\mathfrak{H}=\mathfrak{M}_1$.

We now assume that H_0 and H_1 satisfy the assumption (1) and construct μ, ν etc. in such a way that they satisfy (i), (ii), (a) and (b) in Theorem 2.

Let $U_p=(H_p-i)(H_p+i)^{-1}$ be the Cayley transform of H_p. Then, the assumption (1) implies that $K=(U_1-U_0)U_0^{-1}\in T(\mathfrak{H})$ and it is expressible as

$$K = \sum_{k=1}^{\infty} a_k(\cdot,\ \phi_k)\phi_k,$$

where $(\phi_i,\ \phi_j)=\delta_{ij}$, $|1+a_k|=1$, $|a_k|\neq 0$, and $\sum|a_k|<\infty$. Furthermore, let $H_p=\int x dE_p(x)$ be the spectral resolution of H_p and let $F_p(e)=\int_e(1+x^2)dE_p(x)$ for each bounded Borel set e.

Let now $\mathfrak{C}$ be the closed subspace of $\mathfrak{H}$ spanned by $\{\phi_k\}$ and put

$$\mu(e) = \xi^*F_0(e)\xi, \qquad \nu(e) = \eta^*F_1(e)\eta \tag{4}$$

where ξ and η be given by

$$\xi = \sum_{k=1}^{\infty} \xi_k(\cdot,\ \phi_k)\phi_k, \qquad \eta = \sum_{k=1}^{\infty} \eta_k(\cdot,\ \phi_k)\phi_k, \tag{5}$$

with $\{\xi_k\}$ and $\{\eta_k\}$ being square summable sequences to be determined below. ξ and η are considered to be operators from $\mathfrak{C}$ to $\mathfrak{H}$ so that $\mu(e)\in T_p(\mathfrak{C})$ and $\nu(e)\in T_p(\mathfrak{C})$. We further assume that α and β in (3) have the form

$$\alpha = \sum_{k=1}^{\infty} \alpha_k(\cdot,\ \phi_k)\phi_k, \qquad \beta = \sum_{k=1}^{\infty} \beta_k(\cdot,\ \phi_k)\phi_k \tag{6}$$

with bounded real sequences $\{\alpha_k\}$, $\{\beta_k\}$ and want to determine these sequences so that the relation (3) is true. The source of a reciprocal relation such as (3) is the following reciprocal relation in the operator form:

$$\{1 + K'(U_0 - w)^{-1}\}\{1 - K'(U_1 - w)^{-1}\} = 1, \qquad |w| \neq 1, \tag{7}$$

where we put $K'=U_1-U_0=KU_0$. On the other hand, (4) gives that $\alpha+\phi_\mu(z)=\alpha+i\pi^{-1}\xi^*(U_0+w)(U_0-w)^{-1}\xi$ with $w=(z-i)(z+i)^{-1}$. By using this and the similar relation for ν to express (3) in terms of w and comparing it with (7), we have the following proposition.

PROPOSITION. *If we put* $\xi_k=|a_k|^{1/2}\xi_k'$ *with an arbitrary sequence* $\{\xi_k'\}$ *of complex numbers such that* $0<a\leqq|\xi_k'|\leqq b<\infty$ *for some positive* a *and* b, *and determine* $\{\eta_k\}$, $\{\alpha_k\}$ *and* $\{\beta_k\}$ *successively by the relations*

$$
(8) \qquad \begin{aligned}
2\xi_k\bar{\eta}_k &= -\pi a_k(\bar{a}_k + 1) = \pi\bar{a}_k, \\
\alpha_k &= i\pi^{-1}(1 + 2/a_k)\,|\,\xi_k\,|^2, \\
\beta_k &= i\pi^{-1}(1 + 2/\bar{a}_k)\,|\,\eta_k\,|^2,
\end{aligned}
$$

then μ, ν, α, *and* β *defined by* (4), (5), *and* (6) *satisfy the relation* (3). *Furthermore, it automatically follows that* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *are real and bounded and that* $\sum|\xi_k|^2$, $\sum|\eta_k|^2 < \infty$.

We now construct π_0 and π_1. We can assume without loss of generality that the set of all elements of $\mathfrak{H}$ of the form

$$
u = \sum_{k=1}^{n} u_k^{(p)}(H_p)\phi_k,
$$

with $u_k^{(p)}$ such that $\int|u_k^{(p)}(x)|^2 d\|E_p(x)\phi_k\|^2 < \infty$ forms a dense set in $\mathfrak{H}$ for each $p=0, 1$. (The closure of the above set is independent of p and on its orthogonal complement we have $H_0 = H_1$.) For such a u with $p=0$ we define

$$
(\pi_0 u)(x) = \sum_{k=1}^{n} \xi_k^{-1} u_k^{(0)}(x)(x+i)^{-1}\phi_k
$$

and $(\pi_1 u)(x)$ similarly with ξ replaced by η. Then, π_0 and π_1 can be uniquely extendable to unitary operators from $\mathfrak{H}$ on $L^2(\mu)$ and $L^2(\nu)$, respectively. Now the very relation (8) which ensured the validity of (3) also implies the relation $T\pi_0 = i\pi_1$. Thus, we have the following theorem.

THEOREM 3. *With* μ, ν, α, *and* β *defined in the Proposition and* π_p, $p=0, 1$, *constructed as above, the conditions* (i), (ii), (a) *and* (b) *in Theorem* 2 *hold true. Thus, under the assumption* (1), $W_\pm(H_1, H_0)$ *exists and is given by*

$$
(9) \qquad W_\pm(H_1, H_0) = -i\pi_1^{-1}\Omega_\pm\pi_0
$$

with $\Omega_\pm$ *constructed as in Lemma* 1.

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