CONTINUITY PROPERTIES OF MONOTONE NONLINEAR OPERATORS IN BANACH SPACES

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In a number of recent papers, the writer ([1]–[11]) and G. J. Minty ([13]–[16]) have studied nonlinear functional equations in Banach spaces involving monotone operators satisfying various mild continuity conditions (in particular demicontinuity, hemicontinuity, and vague continuity). In his note [12], T. Kato has studied the inter-relations of these continuity assumptions for monotone operators and shown in particular that every hemicontinuous locally bounded monotone operator G from a Banach space X to its dual X^* is always demicontinuous. The writer has independently obtained this and related results by a slightly different method in connection with his study of multi-valued monotone nonlinear mappings [9]. We present this argument below in §1.

1. Let X be a complex Banach space, X^* the space of bounded conjugate-linear functionals on X, (w, u) the pairing between w in X^* and u in X. Following the notation of [12], $\rightarrow$ denotes strong convergence in X or X^*, $\rightharpoonup$ weak* convergence in X^*.

Let G be a function with domain $D=D(G)\subset X$ and values in X^*. Then:

(1) G is said to be *monotone* if

$$\mathrm{Re}(Gu - Gv, u - v) \geqq 0$$

for all u, v in D.

(2) G is said to be *demicontinuous* if $u_n \rightarrow u$ in D implies $Gu_n \rightharpoonup Gu$.

(3) G is said to be *hemicontinuous* if $u\in D$, $v\in X$ and $u+t_nv\in D$ where $t_n>0$, $t_n\rightarrow 0$, together imply $G(u+t_nv)\rightharpoonup Gu$.

(4) G is said to be *vaguely continuous* if $u\in D$, $v\in X$ and $u+tv\in D$ for $0<t<t_0$ for some $t_0>0$ imply that there exists a sequence $\{t_n\}$ with $t_n>0$ for all n, $t_n\rightarrow 0$ as $n\rightarrow +\infty$ such that $G(u+t_nv)\rightharpoonup Gu$.

(5) G is said to be *D-maximal monotone* if for $u_0\in D$, $w_0\in X^*$, the inequality $\mathrm{Re}(w_0-Gu, u_0-u)\geqq 0$ for all u in D implies that $w_0 = G(u_0)$.

(6) G is said to be *locally bounded* if for any sequence $\{u_n\}$ in D

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with $u_n \to u$ where $u \in D$, we have Gu_n bounded.

(7) D is said to be *quasi-dense* [12] if for each u in D there exists a dense subset M_u of X such that for each $v \in M_u$, $u+tv \in D$ for sufficiently small $t>0$ (the smallness of t depending on v).

THEOREM 1. *If G is D-maximal monotone and locally bounded, then G is demicontinuous.*

PROOF OF THEOREM 1. Let $\{u_n\}$ be a sequence in D with $u_n \to u_0$, $u_0 \in D$. Since $\{Gu_n\}$ is bounded in X^*, to show that $Gu_n \rightharpoonup Gu_0$, it suffices to show that if $Gu_n \rightharpoonup w_0$, then $w_0 = Gu_0$. Let u be any element of D. For each n, $\text{Re}(Gu_n - Gu, u_n - u) \geqq 0$. As $n \to \infty$, $u_n - u \to 0$, $Gu_n - Gu \rightharpoonup w_0 - G_u$. Hence

$$\text{Re}(w_0 - Gu, u_0 - u) \geqq 0.$$

Since G is D-maximal monotone, $w_0 = Gu_0$. Q.E.D.

THEOREM 2. *Suppose that G is monotone and vaguely continuous and that its domain D is quasi-dense. Then G is D-maximal monotone.*

PROOF OF THEOREM 2. Suppose $w_0 \neq Gu_0$. Since D is quasi-finitely dense, there exists v in M_{u_0} such that $u_0 + tv$ lies in D for $0 < t < t_0(v)$ while $\text{Re}(w_0 - Gu_0, v) > 0$.

By the vague continuity of G, we may find a sequence of positive numbers $\{t_n\}$ with $t_n \to 0$ as $n \to \infty$ such that $G(u_0 + t_n v) \rightharpoonup Gu_0$. Then for every n:

$$0 \leqq \text{Re}(G(u_0 + t_n v) - w_0, t_n v)$$

so that since $t_n > 0$,

$$0 \leqq \text{Re}(G(u_0 + t_n v) - w_0, v).$$

Hence

$$0 < \text{Re}(w_0 - Gu_0, v) \leqq \liminf \text{Re}(G(u_0 + t_n v) - Gu_0, v) = 0,$$

which is a contradiction. Q.E.D.

THEOREM 3. *Suppose that G is monotone, locally bounded, and vaguely continuous while D is quasi-dense. Then G is demicontinuous.*

Theorem 3 is a corollary of Theorems 1 and 2 and contains Theorem 1 of [12] as a special case.

REMARK. In Theorem 2 of [12], Kato shows that for finite-dimensional X, local boundedness can be eliminated from the hypothesis of the result corresponding to Theorem 3. This is obviously not the case in infinite dimensional spaces, as one can see from the consideration of unbounded non-negative self-adjoint operators in an infinite dimensional Hilbert space. Whether it is true under the additional hypothesis that $D = D(G) = X$ is an open question. We note that in this case it follows from the argument of Theorem 1 above and a category argument that the set of points in X at which G is continuous includes an open dense set.

REMARK 2. The proof given by Kato for his Theorem 2 in [**12**] actually proves the following:

THEOREM 4. *If X is of finite dimension, G a monotone mapping of X into X^* with quasi-dense domain D, then G is locally bounded.*

Combining this result with Theorem 3 above, we have:

THEOREM 5. *Let X be of finite dimension, G a vaguely continuous monotone mapping from X to X^* with quasi-dense domain D. Then G is continuous.*

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