A GENERALIZATION OF MORSE-SMALE INEQUALITIES[1]

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In this paper we obtain relations between simple periodic surfaces of a vector field on a closed manifold M^n, and the betti numbers of M^n. When X is a gradient vector field of a nondegenerate function on M, the simple periodic surfaces are the critical points of the function and our relations are the Morse Inequalities. For Morse-Smale dynamical systems, the simple periodic surfaces are the critical points and closed orbits and we obtain the inequalities of Smale. In this case, where the periodic surfaces are singularities and closed orbits, we are able to remove Smale's normal intersection condition and replace it by the much weaker condition: there are no cycles of orbits among the periodic surfaces. Consequently, in this context, the need for approximating gradient fields by Morse-Smale systems is eliminated. This is an announcement of the results; detailed proofs will appear elsewhere.

I. Periodic surfaces of vector fields.

DEFINITIONS. Let X be a C^∞ vector field on M. A periodic i surface of X is a submanifold of M, invariant under X which is homeomorphic to T^i, the i dimensional torus. T^0 is a point so a periodic zero surface is a critical point of X; a periodic one surface is a closed orbit and a periodic two surface is a two torus which is the union of trajectories of X.

A simple periodic i surface β of X, of index j is a periodic surface β of M satisfying:

There is a tubular neighborhood N of β wherein X is topologically equivalent to one of the vector fields $Y_1, Y_2 \in \mathfrak{X}(T^i \times R^{n-i})$ defined by

I. $Y_1(\theta, y) = (1, f_1(\theta, y), B_1 y + G_1(\theta_1, y))$ where

$$(\theta, y) = (\theta_1, \cdots, \theta_i, y_1, \cdots, y_{n-i}) \in T^i \times R^{n-i}\ (1, f_1(\theta, y)) \in T^i, f_1 \colon T^i \times R^{n-i} \to 1 \times T^{i-1} \subset T^i,$$

f_1 is C^∞, B_1 is a real $n-i$ matrix having no eigenvalue with zero real part and $G_1 \colon T^1 \times R^{n-i} \to R^{n-i}$ is C^∞, quadratic in y and $G_1(\theta_1, 0) = 0$,

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[1] This is part of my doctoral dissertation, submitted at Berkeley in June, 1963. I wish to thank Professor Diliberto, who was my thesis director and originally suggested this research.

$\theta_1 \in T^1$. The number of eigenvalues of B_1 having real part less than zero is called the index of β.

II. $Y_2(\theta, y) = (f_2(\theta, y), B_2 y + G_2(y))$ where $f_2: T^i \times R^{n-i} \to T^i$ is C^∞, B_2 an $n-i$ matrix having no eigenvalue with zero real part, $G_2: R^{n-i} \to R^{n-i}$ is C^∞, $G_2(0)=0$ and G_2 is quadratic in y. In case II we define the index of β as above, viz., it is the number of eigenvalues of B_2 with real part less than 0.

The α and ω limit sets of X have their usual meaning. If $A \subset M$ we define the stable and unstable sets of A: $W_A^* = \{q \in M/\omega(q) \subset A\}$, $W_A = \{q/\alpha(q) \subset A\}$.

We will see that the index of β is a topological invariant; indeed, we shall show that W_β, the unstable manifold of β, is homeomorphic to $T^i \times R^j$, j the index of β.

Notice that β can be a periodic zero surface, i.e. critical point, of Y_2 but not of Y_1. The closed orbits and higher tori of Y_1 admit a cross section given by $\theta_1 = 1$, yet we place no restriction on X/β when X is of type II. For definitions of topological equivalence and cross section we refer the reader to [5], and details concerning periodic surfaces may be found in the papers by Diliberto [1]. In particular, the question of when a periodic i surface is simple is considered there.

We may now state our main result.

THEOREM 1. *Let $X \in \mathfrak{X}(M^n)$ have a finite number of disjoint simple periodic surfaces $\beta_1, \cdots, \beta_\nu$, satisfying:*

(i) *The limit sets of X are contained in the β_i, i.e., if $x \in M$, $\exists i, j \ni \alpha(x) \subset \beta_i$, $\omega(x) \subset \beta_j$.*

(ii) *There is no closed cycle of orbits among the β_i; more precisely: there is no sequence $i = i_1, \cdots, i_k = i \ni W_{i_j} \cap W^*_{i\,j+1} \neq \emptyset$ for $1 \leqq j \leqq k-1$. Let R_i be the ith betti number of M with respect to a coefficient field. Let $a_j^i =$ the number of periodic i surfaces of index $j-i$, $a_j^i = 0$ if $j < i$. Let*

$$M_q = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} a_{q+i}^{k}, \qquad \binom{k}{i} = \frac{k!}{i!(k-i)!}.$$

Then

$$\begin{aligned} &M_0 \geqq R_0, \\ &M_1 - M_0 \geqq R_1 - R_0, \\ &M_2 - M_1 + M_0 \geqq R_2 - R_1 + R_0, \\ &\qquad\vdots \\ &\sum_{i=0}^{n} (-1)^i M_i = \sum_{i=0}^{n} (-1)^i R_i = (-1)^n \mathfrak{X}, \end{aligned}$$

$\mathfrak{X}$ = *Euler characteristic of M.*

REMARKS. (1) Let f be a C^∞ function on M with nondegenerate critical points $\beta_1, \cdots, \beta_\nu$. Then $X = \operatorname{grad} f$ satisfies the conditions of Theorem 1, and $M_q = a_q^0 =$ the number of critical points of index q. Hence we obtain the usual Morse inequalities [2]. A lemma of Morse ensures us we may find coordinates about a critical point β wherein

$$X(y_1, \cdots, y_n) = (-y_1, \cdots, -y_j, y_{j+1}, \cdots, y_n).$$

Thus each critical point is a simple periodic zero surface and index has the customary meaning. If $\phi_t(p)$ is the solution curve of X through p, then $f\phi_t(p)$ is a strictly increasing function of t or p is a critical point. Now the reader may easily check that X satisfies (i) and (ii).

(2) Suppose X is a Morse-Smale dynamical system [4]. Then X satisfies the conditions of Theorem 1 and $M_q = a_q^0 + a_q^1 + a_{q+1}^1$; Theorem 1 is then Smale's result. The critical points and closed orbits of X are simple periodic zero and one surfaces by definition. One need only check the Smale's normal intersection condition implies condition (ii) of Theorem 1, $\exists$ no closed sequence of solution curves. We leave this to the reader.

The following lemma allows us to obtain a decomposition of M into closed invariant subspaces. We state it in complete generality. However, for its application, A_i will be a periodic k surface and W_{A_i} the image of a 1-1 continuous map from $T^k \times R^j$ onto W_{A_i}.

LEMMA 1. *Let $X \in \mathfrak{X}(M)$ and $A_1, \cdots, A_\nu$ be subsets of $M \ni$*

(i) *the limit sets are the A_i; more precisely, if $p \in M \;\exists i, j \ni \alpha(p) \subset A_i$, $\omega(p) \subset A_j$,*

(ii) *there is no cycle of solution curves among the A_i, i.e., there is no sequence $i_1, \cdots, i_l \ni W_{i_j} \cap W^*_{i_{j+1}} \neq \emptyset$ for $1 \leq j \leq l-1$ and $W_{i_l} \cap W^*_{i_1} \neq \emptyset$,*

(iii) *if $\partial W_\gamma \cap W_\beta \neq \emptyset$ then $\exists$ sequence $\gamma = i_1, \cdots, i_k = \beta$, and $W_{i_j} \cap W^*_{i_{j+1}} \neq \emptyset$ for $1 \leq j \leq k-1$.*

Let $L_0 = U\{W_j / \partial W_j = \emptyset\}$ and $L_i = U\{W_j / \partial W_j \subset L_{i-1}\}$. Then $L_0 \subset L_1 \subset \cdots \subset L_k = M$ for some k.

PROOF. First we show $L_0 \neq \emptyset$, i.e. $\exists A_i \ni \partial W_{A_i} = \emptyset$, which means $W_{A_i} = A_i$ by (i), hence it is enough to show $\exists A_i \ni$ no orbit came from A_i. Consider A_1. If $\exists$ no $p \in M - A_1 \ni \alpha(p) \subset A_1$ then we are done. Otherwise let p_1 satisfy $\alpha(p_1) \subset A_1$. By (i) $\exists j$, $j = 2$ say, $\ni \omega(p_1) \subset A_j$. We cannot have $j = 1$ by condition (ii). Now either $\exists p_2 \in M - A_2 \ni \alpha(p_2) \subset A_2$ or $A_2 \in L_0$. If the former case occurs, let $\omega(p_2) \subset A_k$ for some k. Clearly we cannot have $k = 1$ or 2 by condition (ii); take

$k=3$ say. Continuing in this way we obtain a $k \ni A_k = W_{A_k} \in L_0$. Hence $L_0 \neq \emptyset$.

Now suppose we have shown $L_i \neq \emptyset$, yet $L_{i+1} \neq M$. Then we would like to know $L_{i+1} - L_i \neq \emptyset$. Let $A_{i_1}, \cdots, A_{i_p}$ be the A's in $M - L_i$. We can assume every orbit leaving A_{i_1} goes to some A in L_i by reasoning as above. We claim $\partial W_{A_{i_1}}$ is then in L_i. Suppose not, then $\partial W_{A_{i_1}}$ meets W_{A_j} for $j \in \{i_2, \cdots, i_p\}$. By condition (iii) $\exists$ sequence of orbits going from A_{i_1} to A_j. But every orbit leaving A_{i_1} goes into L_i which is invariant under X, hence no orbit leaves L_i to go to A_j which yields the desired contradiction.

LEMMA 2. *Let X satisfy the conditions of Theorem 1 and β a simple periodic i surface of index J: Then W_β is homeomorphic to $T^i \times R^j$.*

LEMMA 3. *Let X and β be as in Lemma 2. Suppose y^i is a sequence in $M \ni y^i \notin W_\beta$ and $\lim_i y^i = y \in \beta$. Then $\exists x \in W_\beta^*$, $x \notin \beta$ and a sequence $t_k \ni \lim_k \phi_{t_k}(y^{n_k}) = x$.*

SKETCH OF PROOF. For convenience we assume $i=0$. Since the index of β is j, and the conclusion of Lemma 3 is invariant under topological equivalence, we may assume that locally X is the field

$$X(p_1, \cdots, p_n) = (-p_1, \cdots, p_j, p_{j+1}, \cdots, p_n)$$

Clearly $\phi_t(p) = (e^{-t}p_1, \cdots, e^{-t}p_j, e^{+t}p_{j+1}, \cdots, e^{+t}p_n)$,

$$W_\beta^* = \{(p_1, \cdots, p_n)/p_1 = p_2 = \cdots = p_j = 0\},$$

$$W_\beta = \{(p_1, \cdots, p_n)/p_{j+1} = \cdots = p_n = 0\}.$$

Since $y^i \notin W_\beta$, for each i, $\exists k \geqq j \ni y_k^i \neq 0$. Then $\exists t_k \ni e^{-t_k} = \max\{|y_{j+1}^i|, \cdots, |y_n^i|\}$.

Consider $\phi_{t_k}(y_1^i) = (e^{-t_k}y_1^i, \cdots, e^{-t_k}y_j^i, e^{t_k}y_{j+1}^i, \cdots, e^{t_k}y_n^i)$. For each i, one of the last $n-j$ coordinates of $\phi_{t_k}(y^i)$ is ± 1 and the others are less than one in absolute value. The first j coordinates converge to 0 as $i \to \infty$ since $y^i \to 0$ and $e^{-t_k} \to 0$. Hence some subsequence of $\phi_{t_k}(y^i)$ converges to a point in $W_\beta^* - \beta$.

LEMMA 4. *If $\partial W_\gamma \cap W_\beta \neq \emptyset$ then $\exists$ sequence $\gamma = i_1, \cdots, i_k = \beta \ni W_{i_j} \cap W_{i^*_{j+1}} \neq \emptyset$ for $1 \leqq j \leqq k-1$.*

This follows easily from Lemma 3 and condition (ii).

Now it is clear that under the hypothesis of Theorem 1, there is a decomposition of M into closed subspaces L_i with $L_{i+1} - L_i =$ disjoint union of W_j's.

LEMMA 5. *Let* $\phi \subset L_1 \subset L_2 \subset \cdots \subset L_k = M$ *be closed subspaces and* H^* *a cohomology theory* $\ni$ $\dim H^k(L_i, L_{i-1}) < \infty$. *Let* $S_q = \sum_{i=1}^k \dim H^q(L_i, L_{i-1})$ *and* $R_q = \dim H^q(M; F)$, *F a field. Then* S_q *and* R_q *satisfy the Morse inequalities.*

A proof of Lemma 5 may be found in [3].

To complete the proof of Theorem 1 it suffices to show

LEMMA 6. $M_q = S_q$.

PROOF. We evaluate S_q in Čech cohomology with H_k^q cohomology with compact support:

$$\begin{aligned} S_q &= \sum_{i=1}^{k} \dim H^q(L_i, L_{i-1}) = \sum_{i=1}^{k} \dim H_k^q(L_i - L_{i-1}) \\ &= \sum_{i=1}^{k}\Bigg(\sum_{j/\omega_j \subset L_i - L_{i-1}} \dim H_k^q(W_j)\Bigg) = \sum_{\text{all } \beta_j} \dim H_k^q(W_j) \\ &= \sum_{i=0}^{n}\Bigg(\sum_{j=0}^{n-i} \dim H_k^q(T^i \times R^j)\cdot a_{i+j}^i\Bigg) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} \dim H^{q-j}(T^i)\cdot a_{i+j}^i \\ &= \sum_{i=0}^{n}\sum_{i=0}^{n-i}\binom{i}{q-j}\cdot a_{i+j}^i = \sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i} a_{q+i}^k = M_q. \end{aligned}$$

II. **Periodic surfaces of diffeomorphisms.** Let L be a diffeomorphism of M^n. A k dimensional submanifold P^k of M^n is called a $k+1$ dimensional periodic surface of L of period l if P^k is homeomorphic to T^k and $L^l(P^k) = P^k$, l minimal with respect to this property.

We say P^k is elementary if $\exists$ neighborhood N of $P^k \ni L^l/N$ is topologically conjugate to a diffeomorphism

$$\eta: T^k \times R^{n-k} \to T^k \times R^{n-k},$$
$$\eta(\theta, y) = (\eta_1(\theta, y), \eta_2(y)),$$

η_2 a diffeomorphism of R^{n-k} leaving the origin fixed whose Jacobian at the origin has no characteristic exponent equal to one.

THEOREM 2. *Let* L *be a diffeomorphism of a closed manifold* M^n *satisfying*:

(i) *There are a finite number of disjoint elementary periodic surfaces of* L, $P_1, \cdots, P_m \ni W_{p_i}^*$ *is a submanifold of* M *for each* i.

(ii) *If* $x \in M$, $\exists$ $i, j \ni \alpha(x) \subset P_i$, $\omega(x) \subset P_j$.

(iii) *There is no sequence* $P_{i_1}, \cdots, P_{i_k} \ni P_{i_k} = P_{i_1}$, *and* $W_{P_{ij}} \cap W^*_{P_{ij+1}} \neq \emptyset$ *for* $1 \leqq j \leqq k-1$.

Let a^i_j be the number of P's whose stable manifold is of dimension $i+j$. Then the numbers

$$M_q = \sum_{k=0}^{n} \sum_{i=0}^{k} \binom{k}{i} a^k_{q+i} \text{ and } R_q = \dim H^q(M; F),$$

satisfy the Morse inequalities.

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COHOMOLOGY OF CYCLIC GROUPS OF PRIME SQUARE ORDER

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1. **Introduction.** Let G be a cyclic group of order p^2, p a prime, and let U be its unique proper subgroup. If A is any G-module, then the four cohomology groups

$$H^0(G, A) \quad H^1(G, A) \quad H^0(U, A) \quad H^1(U, A)$$

determine all the cohomology groups of A with respect to G and to U. We have determined what values this ordered set of four groups takes on as A runs through all finitely generated G-modules.

2. **Methods of proof.** First we show that every finitely generated G-module has the same cohomology as some finitely generated R-torsion free RG-module, where R is the ring of p-adic integers. Be-