ON THE RELATIVE THEORY OF TAMAGAWA NUMBERS

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This note is an outline of some of the author's recent work on the relative theory of Tamagawa numbers of semisimple algebraic groups. The details and applications will be published elsewhere.

Let k be an algebraic number field of finite degree over $\boldsymbol{Q}$, let G be a connected semisimple algebraic group defined over k. G admits one and only one simply connected covering $(\tilde{G}, \pi)$ defined over k (except for isomorphisms over k). Denote by $\mathfrak{g}$ the Galois group of $\bar{k}/k$, where $\bar{k}$ means the algebraic closure of k. Then the finite commutative group Ker π obtains a structure of a $\mathfrak{g}$-module. Our purpose is to express the Tamagawa number of the isogeny π, i.e. the number $\tau(\pi)=\tau(G)/\tau(\tilde{G})$, in terms of some invariants of the module Ker π. For the notion of the Tamagawa number, see [6], [2]. We denote by k_v the completion of k at a place v of k and use $v=\mathfrak{p}$ if v is nonarchimedean. We also use the standard notation in the Galois cohomology of algebraic groups [1]. We say that an algebraic group A defined over k is of type (K) if $H^1(k_{\mathfrak{p}}, A)=0$ for all $\mathfrak{p}$ and the map $H^1(k, A) \to \prod_v H^1(k_v, A)$ is injective. Kneser has conjectured that every simply connected semisimple group defined over k is of type (K) and has verified it for many classical groups [4].

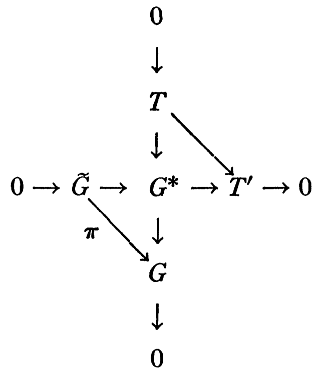
THEOREM. *Let $(\tilde{G}, \pi)$ be the universal covering of a connected semisimple algebraic group G defined over k. Assume that $\tilde{G}$ is of type* (K). *Then the ratio of Tamagawa numbers: $\tau(\pi)=\tau(G)/\tau(\tilde{G})$ is a rational number which can be expressed as*

$$\tau(\pi) = \frac{[\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ker}\ \pi, \bar{k}^*)]}{[\mathrm{Ker}(H^2(k, \mathrm{Ker}\ \pi) \to \prod_v H^2(k_v, \mathrm{Ker}\ \pi))]}, \tag{1}$$

where [*] *means the order of a finite group* * *and* $\mathrm{Hom}_{\mathfrak{g}}$ *means the (continuous)* $\mathfrak{g}$*-homomorphisms.*

REMARK. A. Weil has conjectured that $\tau(\tilde{G})=1$ [5]. Thus, if the conjectures of Weil and Kneser for simply connected groups and true, the right-hand side of (1) describes $\tau(G)$ for any connected semisimple group G. We have an example of a semisimple group G for which $\tau(G) \notin \boldsymbol{Z}$ [3]; the Tamagawa number depends in general on the field of definition of groups.

OUTLINE OF PROOF. The $\mathfrak{g}$-module Ker π can be imbedded in a torus of type $T=(R_{K/k}(G_m))^r$. One gets a commutative diagram with exact row and column:



where $G^* = (\tilde{G} \times T)/\text{Ker } \pi$ with the diagonal imbedding of Ker π. By computing an integral over G^*_A (the adelization) in two ways as Weil has done for classical groups [6], we can reduce our problem to the isogeny of tori: $T \to T'$. For tori, the Tamagawa number can be written in terms of Galois cohomology [3]. After some cohomological considerations one arrives at the result under the condition (K).

References

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