A LOCALLY COMPACT SEPARABLE METRIC SPACE IS ALMOST INVARIANT UNDER A CLOSED MAPPING

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For a given mapping (continuous transformation) f of a topological space X onto a topological space Y it has always been of interest to determine what properties of X carry over to Y. Under the hypothesis that f is a closed mapping it is known that normality [1], and paracompactness [5] are invariants. If X is metric and f is closed then as a consequence of results of Vaĭnšteĭn [2], Whyburn [1], and Stone [4], it is known that Y is weakly separable if and only if each point inverse has a compact frontier. From [1] we obtain the result that if X is perfectly separable, f is closed and Y is weakly separable then Y is a separable metric space. Let f be a closed mapping of a locally compact separable metric space X onto a topological space Y, or, equivalently, let G be an upper semi-continuous decomposition of X into closed sets. We say a set S is a *scattered* set if every subset of S is closed. Our results show that Y (or the decomposition space M) minus a scattered set S, which has at most a countable number of points, is also a locally compact separable metric space. The techniques of proof are standard and will not be included here.

THEOREM 1. *Let G be an upper semi-continuous decomposition of a locally compact separable metric space into closed sets. Let F be the union of the noncompact elements of G, M the decomposition space determined by G, and ϕ the natural mapping of X onto M. The following are valid.*

(i) *F is a closed set.*

(ii) *For an arbitrary compact set K only a finite number of elements of G in F can intersect K.*

(iii) *F contains at most countably many elements of G.*

(iv) *The union of any subcollection of elements of G in F is a closed set.*

(v) *M is weakly separable at y if and only if the frontier of $\phi^{-1}(y)$ is compact (Stone [4]).*

(vi) *If $\{g_n\}$ is a convergent sequence of compact elements of G with a nonempty limiting set h, then the set $K = \bigcup g_n \cup h$ is compact.*

Let F' be the subset of F composed of the union of the elements that do not have a compact frontier. The set F' is closed by (iv) and $S=\phi(F')$ is closed and is at most a countable set of points. Using this notation we have:

THEOREM 2. *The set $M-S$ is a locally compact separable metric space, where S is a scattered set having at most a countable number of points.*

A lower semi-continuous decomposition of a metric space X has the property that if $\{g_n\}$ is a converging sequence of elements of G with a nonempty limiting set h then h is an element of G. For an open mapping replace the g_n by point inverses. By (vi) of Theorem 1 it is easy to see how to obtain the following theorem which is given in a slightly stronger form by Wallace [3] and Whyburn [6].

THEOREM 3. *If G is both an upper semi-continuous and a lower semi-continuous decomposition of a locally compact connected separable metric space X, then all the elements of G are compact.*

To see that S can be infinite consider the decomposition of the plane into the vertical lines whose equations are $x=n$, n an integer, and the individual points not on these lines.

Added in proof. Separability can be omitted in the hypotheses provided that the countability of F and S is excluded from the conclusions.

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