

Fourier analysis on groups. By Walter Rudin. Interscience Tracts in Pure and Applied Math., No. 12. Wiley, New York, 1962. 9+285 pp. \$9.50.

This work has come in season.

First, it fills a gap between "classical" harmonic analysis (as represented, for example, in Zygmund's book) and "abstract" harmonic analysis (as in Loomis's book). Fourier analysis on (abelian) groups introduces itself in a natural way if one wants to unify the methods which are used in studying ordinary trigonometric series and Fourier integrals, Fourier series or integrals with several variables, almost periodic functions, Taylor and Dirichlet series. It is still more important if one wants to look at the general concepts of Banach algebras, like homomorphisms, symbolic calculus, structure of ideals or subalgebras, in the very field from which the theory originated: the convolution algebra of summable functions on a locally compact abelian group G , the convolution algebra of bounded measures on G , and the algebras of their Fourier transforms (respectively denoted $L^1(G)$, $M(G)$, $A(\Gamma)$, $B(\Gamma)$ through the book and through our review; Γ is the dual group of G). The primary concern of the book is to show which tools of analysis are useful for problems which arise from algebraic considerations.

A second reason to welcome the book is that it provides the reader with much new material. It deals mainly with problems whose solutions were not known five or six years ago. Following the order of the book, we find, after two introductory chapters:

(Chapter III) The problem of finding all idempotent measures on G , i.e. all idempotent elements in $M(G)$ or $B(\Gamma)$, or, equivalently, all sets S in Γ whose characteristic function belongs to $B(\Gamma)$. The problem was investigated by Helson (G =the circle T), Rudin ($G=T^n$), and finally P. J. Cohen. To be brief, the answer is that the only sets S are the obvious ones. But the proofs are not easy, and they are linked with some old and difficult problems like Littlewood's conjecture, that

$$\int_{-\pi}^{\pi} |e^{in_1x} + \dots + e^{in_kx}| dx > c \log k$$

(c absolute constant; n_1, \dots, n_k integers).

(Chapter IV) The problem of homomorphisms of group algebras. Which are the homomorphic mappings from $L^1(G)$ to $L^1(G')$, or, more generally, from $L^1(G)$ to $M(G')$? In other words, which are the $Y \subset \Gamma'$ and the mappings $\phi: Y \rightarrow \Gamma$ such that $f \in A(\Gamma)$ implies $f(\phi) \in B(\Gamma')$? The first study in this direction is due to Beurling and Helson

($\Gamma = \Gamma' = R$), and the last one to P. J. Cohen. When $\Gamma = \Gamma' = R$, ϕ must be a linear function (and that solves a problem asked by Paul Levy in 1934). In the general case, there is an analogous answer. An interesting byproduct of Cohen's work is the proof of a conjecture of Rudin: each automorphism of $M(G)$ carries $L^1(G)$ onto $L^1(G)$.

(Chapter V) How do Fourier transforms of measures carried by "thin" sets behave? That is a topic one can look at from different points of view. Rudin is mainly interested in sets which generalize, in one way or another, independent countable closed sets. The existence of independent perfect sets gives an easy proof of the asymmetry of $M(G)$ when G is not discrete (Williamson) and it is basic in further investigations (Hewitt-Kakutani, Rudin). Kronecker sets are defined as sets E such that each continuous function on E , of absolute value 1, is uniformly approximable by characters. Helson sets on Γ are compact sets E such that each continuous function on E is the restriction of an $f \in A(\Gamma)$. Sidon sets on a discrete Γ are sets E such that each bounded function on E (resp., tending to 0 at infinity) is the restriction of an $f \in B(\Gamma)$ (resp. $A(\Gamma)$). Equivalent definitions are discussed.

(Chapter VI) Which are the functions F which operate on Fourier transforms in the sense that, whatever may be $f \in A(\Gamma)$, with range in the domain of F , $F(f) \in A(\Gamma)$? Paul Levy had asked the question in 1934, while giving the partial answer known as Wiener-Levy theorem (the original statement was made when $\Gamma = R$ or T): the analytic functions (mapping 0 on 0 if Γ is not compact) operate on $A(\Gamma)$. The main result explained in the chapter is that, roughly speaking, the converse is true. Functions which operate on $B(\Gamma)$ are also considered: roughly speaking, they have to be entire functions. The main references are Katznelson's thesis and the paper of Helson-Kahane-Katznelson-Rudin. Results are given on functions which operate on $A(E)$, the algebra of restrictions to a compact set E of $f \in A(\Gamma)$; this particular topic deserves further investigations.

(Chapter VII) Which are the closed ideals in $L^1(G)$? Is each closed ideal I in $L^1(G)$ (or $A(\Gamma)$) the intersection of the maximal ideals which contain I ? In the case $G = R^3$, this was disproved by L. Schwartz and in the general case, for any noncompact G (or nondiscrete Γ), by Malliavin. Together with Schwartz's and Malliavin's fundamental examples, a series of results is discussed, giving a good part of the information available in the literature when the book was published. In particular, several examples of "sets of spectral synthesis" (including Herz's example: the classical triadic Cantor set) are exhibited. This chapter contains several elaborations of previous results; in

particular, Rudin constructs a closed ideal in $A(\Gamma)$ which is not self-adjoint, an interesting refinement of Malliavin's construction.

(Chapter VIII) How to extend properties of Taylor's series, or analytic functions in an open disc, when the group of integers Z is replaced by another group? The question has been investigated by Arens, Helson, Hoffman, and Singer, when Z is replaced by an ordered group. In the case of Z^2 , for example, a Taylor series will be replaced by a Fourier series whose coefficients vanish in a half plane. The fundamental work concerning the L^2 -theory is due to Helson and Lowdenslager, who succeeded in extending classical theorems of Szegő and Beurling. A theorem of the author, on Paley sequences, is given in this general context, as well as the most important results about conjugate functions (L^p -theory, starting from Helson's extension of M. Riesz's theorem).

(Chapter IX) What is the structure of the closed subalgebras in $L^1(G)$ or $A(\Gamma)$? A complete answer to the question, even when Γ is discrete, is out of range (let us mention that the nice conjecture on p. 231 has been disproved since the book was published). Results of Wermer and Simon on maximal subalgebras are discussed, as well as the characterization of those groups Γ such that $A(\Gamma)$ contains no proper closed separating subalgebra ("Stone-Weierstrass property"), discovered by Katznelson and the author.

The mere enumeration of the topics shows that the book is rich in material. Nevertheless, it is a pleasure to read. Two chapters (basic theorems in Fourier analysis, and structure of locally compact abelian groups) and appendices at the end, are intended to make it "self-contained," at the level of a graduate student. In the main part of the book, all the theorems are proved, and all proofs are complete, as far as the reviewer could notice. The author is known to be an excellent expositor, and he proves it once more by providing a considerable amount of information in less than three hundred pages without giving anywhere the impression of hurrying or pressing the reader. The printing is very good, and the book is as good looking as it is thorough.

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Foundations of differential geometry. By Shoshichi Kobayashi and Katsumi Nomizu. Interscience Tracts in Pure and Applied Math., No. 15. John Wiley and Sons, Inc., New York, 1963. 11+329 pp. \$15.00.

Differential geometry has radically changed in recent years. An approach based on the theory of differential manifolds has replaced