VECTOR LATTICES OF SELF-ADJOINT OPERATORS

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1. **Introduction.** A number of authors have dealt with the relations between order and commutativity in operator algebras [**1**; **2**; **3**]. Their results, together with Kadison's characterization [**4**] of factors as anti-lattices, lead one to the belief that commutativity and lattice structure are synonomous concepts. A closer look at these phenomena, however, reveals a number of discrepancies, as we shall presently see from several examples.

We have therefore singled out a special kind of lattice structure and we proceed to show that if the operators in a linear space of self-adjoint (s.a.) operators have this structure, then they form a commuting set. As a consequence of this, we are able to place the results of [**1**; **2**; **3**] in a more coherent framework; the "commutativity-from-order" theorems in these three papers are very elementary (but non-obvious) consequences of our single result; the proof of our central theorem is quite geometrical and provides more insight into the mechanisms relating order and commutativity. Proofs will appear elsewhere.

2. **Order properties.** We first fix some notation. If a is a s.a. operator (bounded, except in §5), we set $|a| = (a^2)^{1/2}$ and define $a \wedge b = \frac{1}{2}(a+b-|a-b|)$.

Now let V be a linear subspace of the space of all s.a. operators and consider the following properties:

(A) If $a \in V$, then $|a| \in V$.

(L) The set V^+ of positive operators in V lattice orders V.

(T) $|a+b| \leqq |a| + |b|$, for all $a, b \in V$.

(P) $a \wedge b \geqq 0$ for $a, b \in V^+$.

(R) For $0 \leqq z \leqq a+b$ with $a, b, z \in V^+$, there exist $u, v \in V$ with $z = u+v$, $0 \leqq u \leqq a$ and $0 \leqq v \leqq b$. (The Riesz decomposition property.)

(SQ) $a^2 \leqq b^2$ for $a, b \in V$ with $-b \leqq a \leqq b$.

(J) $\{a, b\} \geqq 0$ for $a, b \geqq 0$ in V (here $\{a,b\} = \frac{1}{2}(ab+ba)$ is the *Jordan product*).

(O) $a^2 \leqq b^2$ for $a, b \in V$ with $0 \leqq a \leqq b$.

(J$^+$) $\{a, b\} \geqq 0$ for $a, b \in V$ with $0 \leqq a \leqq b$.

From elementary computations, we obtain

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LEMMA 1. *If V is a commuting set, then properties* (T), (P), (SQ), (J), (O) *and* (J^+) *hold. Moreover, we have the following implications*: (J)$\Leftrightarrow$(SQ)$\Leftrightarrow$[(O) *and* (T)] *and* (T)$\Leftrightarrow$(P), *provided that* (A) *holds*; *also* (O)$\Leftrightarrow$(J^+) *and* (L)$\Rightarrow$(R).

DEFINITION. A linear space V of s.a. operators is called a *special vector lattice* if (A) and (L) hold for V and if, in addition, $|a| = \sup(a, -a)$, where $|a| = (a^2)^{1/2}$ and "sup" denotes the least upper bound in the lattice order induced on V by V^+.

REMARK. It is easy to show that a linear space V of s.a. operators is a special vector lattice if and only if (A) and (T) hold.

EXAMPLE 1. If V is any commutative linear space of s.a. operators for which (A) holds, then V is a special vector lattice.

EXAMPLE 2 (SHERMAN [1]). Consider the plane of operators $\alpha a+\beta b$, where α and β are real and

$$a = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\alpha a+\beta b \geqq 0$ if and only if $\alpha \geqq 0$ and $\beta \geqq 0$, the plane satisfies (L); moreover, all operators on this plane commute. However it is easily checked that $|a-b|$ does not lie on this plane, so the lattice is not special.

EXAMPLE 3. Consider the plane of operators $\alpha a+\beta b$, where α and β are real and

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since $\alpha a+\beta b \geqq 0$ if and only if $\alpha \geqq 0$ and $\beta \geqq 0$, this plane satisfies (L). It is easy to check directly that $|a-b|$ does not lie on this plane, and hence the lattice is not special. By construction, the plane is noncommutative.

3. Special vector lattices.

The fundamental fact behind all of the commutativity theorems mentioned in the introduction is

THEOREM 1. *Every special vector lattice of s.a. operators is commutative.*

Of course a special vector lattice need not be an algebra; however, we can give an interesting simple condition which will guarantee that this is the case.

THEOREM 2. *Let V be a special vector lattice and let A be its uniform closure. Let* 1 *denote the identity operator and suppose that* $1 \wedge a \in A$ *for* *each* $a \geqq 0$ *in* V *(Stone's "measurability condition"). Then* A *is an algebra of s.a. operators.*

Theorem 2 gives us a better idea as to why the (nonspecial) vector lattice in Example 2 is not an algebra even though it is uniformly closed, contains the identity and is commutative.

SKETCH OF THE PROOF OF THEOREM 1. First it is shown that any special vector lattice V is a pre-(M)-space (see [5]) under the operator norm, so that V is isometric and linear lattice isomorphic to a sublattice of a $C(X)$, X compact. Next we consider the set L of s.a. operators which are strong limits (i.e. in the strong operator topology) of norm bounded nets from V. Using the fact that the operation $a \to |a|$ is strongly continuous on norm bounded sets, we show that L is again a special vector lattice. Let P denote the positive part of the norm unit ball in V. Then the strong closure (=weak closure, since P is convex) of P, call it Q, is compact in the weak operator topology and lies in L. From the functional representation of L as a pre-(M)-space, we rapidly conclude that the extreme points of Q form a commutative set. The commutativity of L is then immediate.

4. Applications. We now show how Theorem 1 can be applied to generalize and hence yield the existing theorems in which commutativity is deduced from order-theoretic hypotheses. The following lemmas are all of an elementary nature and no proofs will be indicated.

LEMMA 2. *Suppose* $a, b \geqq 0$ *have a greatest lower bound* g *relative to some linear space* V *of s.a. operators containing them, and that* $ab=0$. *Then* $g=0$.

LEMMA 3. *Let* $a, b \geqq 0$ *have a greatest lower bound* g *relative to a linear space* V *of s.a. operators, which contains* $|a-b|$ *as well as* a *and* b. *Then* $g=a \wedge b$ *and hence, in particular,* $a \wedge b \geqq 0$.

COROLLARY 1. *Let* V *be a linear space of s.a. operators which satisfies conditions* (A) *and* (L). *Then* V *is a special vector lattice.*

COROLLARY 2 (SHERMAN'S THEOREM). *If the s.a. operators in a* C^**-algebra* A *are lattice ordered by* A^+, *then* A *is commutative.*

LEMMA 4. *Let* V *be a linear space of s.a. operators which has property* (R). *If* $a, b \geqq 0$ *in* V *are such that* $ab=0$, *then* a *and* b *have greatest lower bound zero in* V.

COROLLARY 3. *If* V *is a linear space of s.a. operators which satisfies conditions* (A) *and* (R), *then* V *is a special vector lattice.*

COROLLARY 4 (THEOREM OF FUKAMIYA-MISONOU-TAKEDA). *If A is a C^*-algebra whose set of s.a. operators has property* (R), *then A is commutative.*

LEMMA 5. *Let V be a linear space of s.a. operators in which* (O) *holds. Then for $a, b \geqq 0$ in V with $ab = 0$, a and b have greatest lower bound zero in V.*

COROLLARY 5. *If V is a linear space of s.a. operators which has properties* (A) *and* (O), *then V is a special vector lattice.*

COROLLARY 6 (OGASAWARA'S THEOREM). *If A is a C^*-algebra whose set of s.a. operators has property* (O), *then A is commutative.*

5. **Unbounded operators.** Let A be a von Neumann algebra of type II_1 with a faithful (scalar-valued) trace, i.e., a type II_1 algebra with countably decomposable center or a II_1 factor. Let M be the *-algebra of measurable operators constructed over A. Since each positive operator in M has a unique positive square root in M, we can define $|a| = (a^2)^{1/2}$ as before and repeat the definition of "special vector lattice" verbatim. Let L^1 denote the noncommutative space of summable operators arising from the trace on A.

THEOREM 3. *Every special vector lattice of unbounded s.a. operators which is contained in L^1 is commutative and its trace norm closure is an abstract* (L)*-space* (*also commutative*).

Now suppose that A is merely an AW*-algebra of finite type (dropping the assumption of countable decomposability) and let M denote its associated regular ring in the sense of [6]. Using the extensive and very elegant theory developed by Berberian in [6] we obtain:

LEMMA 6. *Lemmas* 1, 2, 3, 4, 5 *and Corollaries* 1, 3 *and* 5 *remain valid for V a linear subspace of the self-adjoint elements in the regular ring of a finite AW*-algebra.*

We immediately conclude the following facts from a few elementary computations in the context of [6]:

THEOREM 4. *Let V be a special vector lattice of s.a. elements in the regular ring M and assume that V has the property:*

For each $a \in V^+$, V contains the "spectral family" of a. Then V is commutative.

THEOREM 5. *Let R be a *-subalgebra in the regular ring of a finite AW*-algebra and suppose that R equals its double commutant* (*relative to the regular ring*). *Then any one of the properties*: (L), (T), (P), (R), (SQ), (J), (O) *or* (J^+) *implies the commutativity of R.*

CONJECTURE. Every special vector lattice of self-adjoint elements in the regular ring of a finite AW*-algebra is commutative.

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REMARK ON MY PAPER "SIMULTANEOUS APPROXIMATION AND ALGEBRAIC INDEPENDENCE OF NUMBERS"

BY W. M. SCHMIDT

It has been pointed out to me that a result very similar to the one proved in my paper was obtained by O. Perron, *Über mehrfach transzendente Erweiterungen des natürlichen Rationalitätsbereiches*, Sitzungsberichte Bayer. Akad. Wiss. H2 (1932), 79–86.

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INTEGRAL NORMS OF SUBADDITIVE FUNCTIONS

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It is known that certain integral norms for positive, measurable, subadditive functions of a single variable are comparable (cf. [1]; also [3; 4] for less complete results). This fact was shown to have

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