RESEARCH PROBLEMS

29. Richard Bellman: *Number theory.*

Consider the linear differential operator of order n,

$$L_n(D) = D^n + p_1(t)D^{n-1} + \cdots + p_{n-1}(t)D + p_n(t) \tag{1}$$

where the $p_i(t)$ are polynomials in t with coefficients which are integers modulo p, a prime. The symbol D is the derivative d/dt with the usual properties.

The operator will be said to be *reducible* if we can write

$$L_n(D) = L_m(D)L_r(D)(\text{modulo } p) \tag{2}$$

where $L_m(D)$ and $L_r(D)$ are linear operators of the same type of orders m and r respectively, $m+r=n$; and *irreducible* otherwise.

Let the degrees of $p_1(t), \cdots, p_n(t)$ as polynomials in t be respectively $d_1, d_2, \cdots, d_n$, and let us call $L_n(d)$ of type $[n; d_1, d_2, \cdots, d_n]$. Can one obtain expressions for the number of operators of type $[n; d_1, d_2, \cdots, d_n]$ which are irreducible modulo p?

30. Richard Bellman: *Number theory—generalized cyclotomic sums.*

Can one obtain results for multidimensional cyclotomic sums of the form

$$S(x, \omega_1, \omega_2) = \sum_{m,n=0}^{p-1} \omega_1^m \omega_2^n \exp(2\pi i x a^m b^n/p) \tag{1}$$

where ω_1 and ω_2 are $(p-1)$st roots of unity and a and b are primitive roots modulo p, corresponding to those existing for the one-dimensional sums?

More generally, if the sequence $\{u_{m,n}\}$ satisfies *two* linear recurrence relations

$$\begin{aligned} u_{m,n} &\equiv a_1 u_{m-1,n} + b_1 u_{m,n-1} + c_1 u_{m-1,n-1}(p), \\ u_{m,n} &\equiv a_2 u_{m-1,n} + b_2 u_{m,n-1} + c_2 u_{m-1,n-1}(p), \end{aligned} \tag{2}$$

with appropriate periodicity constraints on the boundary sequence $\{u_{0,n}\}$, $\{u_{m,0}\}$, can one obtain results for sums of the form

$$S(x, \omega_1, \omega_2) = \sum_{m,n} \omega_1^m \omega_2^n \exp(2\pi i x u_{m,n}/p),$$

and for the generalized sums where the recurrence relations have the form

$$(3)\qquad \begin{aligned} u_{m,n} &= \sum_{i,j=0}^{k} a_{ij}u_{m-i,n-j}(p), \\ u_{m,n} &= \sum_{i,j=0}^{k} b_{ij}u_{m-i,n-j}(p). \end{aligned}$$

These sums are generalizations of the one-dimensional sums of the form $\sum_m \omega_1^m \exp(2\pi i \operatorname{tr}(\beta\alpha^n)/p$, where α and β are algebraic numbers.

31. Richard Bellman: *Probability theory.*

Let the one-dimensional sequence $\{x_n\}$ be generated by means of the recurrence relation

$$(1)\qquad x_{n+1} = ax_n + g(x_n) + r_n, \qquad x_0 = c,$$

where a and c are real, $\{r_n\}$ is a sequence of independent random variables with known identical distribution functions and $g(x)$ is a feedback control function defined as follows:

$$(2)\qquad \begin{aligned} g(x) &= 0, & |x| &< b, \\ &= k, & x &< -b, \\ &= -k, & x &> b, \end{aligned}$$

where $k > 0$.

What possibilities exist for the limiting distributions of x_n, suitably normalized?

The problem can be generalized by considering multidimensional sequences, continuous time, correlation in the r_n, and a larger number of strata for $g(x)$.

32. Richard Bellman: *Prediction theory.*

Given the information that the sequence $\{x_n\}$ is generated by a relation of the foregoing type, how does one determine the unknown parameters b and k from observation of the process up to time N, and make optimal estimates for the values of x_{N+1}?