relativity which is supported by large numbers of accurate experiments. Even within the theoretical framework of general relativity it can surely not be regarded as unimportant; for example, the principle of equivalence, when combined with symmetry properties and the fact that Newtonian gravitational theory must be a valid approximation, leads directly to the conclusion that the space-time of a massive spherical body at rest must be curved. The other important omission is the subject of variational principles for the gravitational field and other fields which interact with it, and the beautiful and deep relationship discovered by Hilbert and Klein between the invariance properties of the action integral and the existence of differential conservation laws. This theory of the structure of field theories is required if one attempts to understand the energy momentum tensor not as a mere phenomenological description of matter, but as a sum of contributions from other fields (electromagnetic field, electron field, etc.) which are the joint sources of the gravitational field. It is also required if one attempts to construct other gravitational theories of the type of general relativity, which share with it some desirable characteristics, covariance and the existence of conservation laws.

Synge's books belong on the shelf of every serious student of relativity—but they should be flanked by other books on the subject. John Power and John Jameson can be proud to have *Relativity: The general theory* dedicated to them.

ALFRED SCHILD

*Mathematical foundations of quantum statistics.* By A. Y. Khinchin (Translation from 1951 Russian Edition edited by Irwin Shapiro). Graylock Press, New York, 1960. 11+232 pp. $10.00.

Let $\mathfrak{H}$ be the Hilbert space whose unit vectors define the pure states for a quantum mechanical particle. Let $H$ be the energy operator and suppose that this operator has a pure point spectrum with eigenvalues $0 \leqq \epsilon_1 \leqq \epsilon_2 \leqq \cdots$. For each positive integer $N$, let $\mathfrak{H}_N$ denote the $N$ fold tensor product of $\mathfrak{H}$ with itself and let $H_N$ denote the sum

$$H \times I \times \cdots \times I + I \times H \times I \cdots \times I + \cdots + I \times I \times I \cdots H$$

where $I$ is the identity and each product has $N$ factors. Then $\mathfrak{H}_N$ and $H_N$ are the Hilbert space and energy operator for a system of $N$ noninteracting replicas of the particle described by $\mathfrak{H}$ and $H$. Let $\mathfrak{H}_N^S$ denote the closed subspace of $\mathfrak{H}_N$ consisting of all symmetric members of the tensor product and let $\mathfrak{H}_N^A$ denote the closed subspace of $\mathfrak{H}_N$ consisting of all anti-symmetric members of the tensor product.

$\mathcal{H}_N^S$ and $\mathcal{H}_N^A$ are invariant under $H_N$. Let us denote the restrictions of $H_N$ to $\mathcal{H}_N^S$ and $\mathcal{H}_N^A$ respectively by $H_N^S$ and $H_N^A$. If the $N$ particles are "bosons" then the Hilbert space and energy operator turn out to be $\mathcal{H}_N^S$ and $H_N^S$ respectively instead of $\mathcal{H}_N$ and $H_N$. If they are "fermions" then the appropriate space and operator are $\mathcal{H}_N^A$ and $H_N^A$. A system with an indefinite number of identical bosons is described by the direct sums $\sum_{N=0}^{\infty} \mathcal{H}_N^S = \mathcal{H}_{\sim}^S$ and $H_{\sim}^S = \sum_{N=0}^{\infty} H_N^S$. If an observable associated with a single particle is described by the self adjoint operator $B$ in $\mathcal{H}$ then we may construct operators $B_N$, $B_N^S$, $B_N^A$ and $B_{\sim}^S$ in $\mathcal{H}_N$, $\mathcal{H}_N^S$, $\mathcal{H}_N^A$, $\mathcal{H}_{\sim}^S$ by direct analogy with the construction of $H_N$, $H_N^S$, $H_N^A$ and $H_{\sim}^S$.

The fundamental problem considered in this book is that of obtaining information about the probability distributions of observables associated with operators of the form $B_N$, $B_N^S$, $B_N^A$ and $B_{\sim}^S$ when the total energy $E$ of the system is known and $N$ is "large." (Of course the proviso about $N$ only makes sense as it stands in the first three cases.) Now it follows from general principles that these probability distributions may be computed as follows. Let $\mathcal{H}'$ denote the Hilbert space, let $H'$ denote the energy operator and let $P_E$ denote the projection on the subspace of all vectors $\phi$ with $H'(\phi) = E\phi$. Let $F \to Q_F^{B'}$ denote the projection valued measure associated with the self adjoint operator $B'$ by the spectral theorem. Then $F \to \text{Trace}(Q_F^{B'} P_E)/\text{Trace}(P_E)$ is the probability distribution of the observable associated with the operator $B'$ when $E$ is the total energy of the system. Its expected value is $\text{Trace}(B'P_E)/\text{Trace}(P_E)$. The denominators of these expressions are finite since one only considers systems in which no eigenvalue of $H'$ has infinite multiplicity. While these formulae are elegant and concise, their exact evaluation in concrete cases is next to impossible. One is forced to rely upon approximations and the formulation and justification of these approximations is one of the central problems of quantum statistical mechanics. Various solutions have been offered and the present book presents one due to its author which is relatively recent and in some respects more satisfactory than earlier ones. It depends upon using a trick to reduce the problem to one to which the limit theorems of probability theory may be applied. In addition to the parameters $E$ and $N$ one has a parameter $V$ representing the volume of space in which the identical particles are confined. The density with respect to which the eigenvalues $\epsilon_1, \epsilon_2, \cdots$ occur is roughly proportional to $V$. The author's main results are asymptotic formulae for the expected values and dispersions of the relevant probability distributions as $N$, $V$ and $E$ tend to infinity in such a way that their ratios remain fixed. The fact that the dispersions tend to zero furnishes a justification for the practice of identifying expected values with observed values and the limits approached by these expected values lead to formulae consistent with and "explaining" those of phenomenological thermodynamics. In a previous work (English Translation: *Mathematical foundations of statistical mechanics*, New York, Dover, 1949) the author treated classical statistical mechanics along similar lines.

The book has six chapters and six "supplements." Supplements V and VI are translations of articles of the author and did not appear in the Russian original. Chapters IV and V form the central core of the book and contain the author's main argument. Chapters I, II, and III are devoted to preliminary material included to make the book more self contained. In Chapter VI the author shows how to pass from his asymptotic results to the usual formulae of thermodynamics.

Chapter I contains the formulation and proof of the limit theorems which are to be used in Chapters IV and V and is based on work of Gnedenko. Chapter II is a brief introduction to quantum mechanics and Chapter III to the general principles of quantum statistical mechanics. In Chapter IV the fundamental asymptotic formulas are derived for the special case of an indefinite number of identical bosons ($\mathcal{H}' = \mathcal{H}^{S}_{\sim}$). This case is simpler than the others because of the absence of the parameter $N$. Its chief application is to the theory of electromagnetic radiation where the bosons are photons. In Chapter V the other three cases are treated more or less simultaneously.

In Supplement I the author explains how his results may be generalized to systems containing more than one kind of particle. In Supplement IV he shows how the argument of Chapter V may be considerably simplified in one of the three cases. When $\mathcal{H}' = \mathcal{H}_N$ it is possible to eliminate one of the two parameters and use a one dimensional limit theorem.

Supplements II and III are concerned with the relationship between "canonical averaging" and "microcanonical averaging." Let $E_1$, $E_2$, be the eigenvalues of the energy operator $H'$, each value being repeated a number of times equal to its multiplicity. Let $p(T)$ denote the function defined by the Dirichlet series $\sum_{j=0}^{\infty} \exp(-E_j/T)$. Let us suppose that instead of knowing the total energy of our system we know only its expected value $E$ and that it takes on the value $E_j$ with probability $\exp(-E_j/T)/p(T)$. For systems of physical interest the value of the parameter $T$ is uniquely determined by $E$ and the equation

$\sum_{j=0}^{\infty} E_j \exp(-E_j/T)=p(T)E$. Then the probability distribution described above becomes $F\rightarrow\text{Trace}(Q_F^{B'} \exp(-H'/T))/p(T)$ and the corresponding expected value becomes $\text{Trace}(B' \exp(-H'/T))/p(T)$. The expected value obtained by assuming that the energy has a definite value is called a microcanonical average. That obtained by assuming that the energy is statistically distributed as just indicated is called a canonical average. In Supplement III the results of Chapter V are used to show that when $N$ is large the microcanonical average may be replaced by the canonical one with a small relative error. This is important because canonical averages are much easier to compute than microcanonical ones. Because canonical averages are so much easier to compute it would be convenient if one could justify using them at the outset instead of the microcanonical ones. One such justification consists in showing that the energy of a "small" component of a "large" system is approximately canonically distributed even when the energy of the whole system has a fixed value. In Supplement II the author gives a proof of this fact by his methods.

In all of the above it has been assumed that the observables in question are what the author calls "sum functions"; that is are described by operators of the form $B_N$, $B_N^A$, $B_N^S$ or $B_{\sim}^S$. In Supplement V he shows that one of his basic results—that relative dispersions tend to zero as $N$ tends to infinity—is valid for a wider class of observables. Roughly speaking the class consists of those which are symmetric functions of the energies of the individual particles. These symmetric functions are required in addition to have a mild regularity property. Supplement V also contains a generalization of the theorem of Supplement II. In this generalization the hypothesis (implicit in microcanonical averaging) that all pure states with the same energy have the same probability is somewhat weakened. In Supplement VI both of the main results of Supplement V are proved for classical statistical mechanics.

The book is written in a clear and pleasing style and the reviewer noted no misprints. However the introductory chapter on quantum mechanics is somewhat inadequate in that the correspondence between self adjoint operators and observables is presented in an oversimplified form. No account is taken of the fact that self adjointness must be defined with some delicacy for unbounded operators.

In sum the author has given us an introduction to quantum statistical mechanics written for mathematicians which establishes certain fundamental results in a more rigorous manner and under weaker hypotheses than had been done previously.

GEORGE W. MACKEY