VECTOR FIELDS ON SPHERES

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Let us write $n=(2a+1)2^b$, where a and b are integers, and let us set $b=c+4d$, where c and d are integers and $0 \leqq c \leqq 3$; let us define $\rho(n)=2^c+8d$. Then it follows from the Hurwitz-Radon-Eckmann theorem in linear algebra that there exist $\rho(n)-1$ vector fields on S^{n-1} which are linearly independent at each point of S^{n-1} (cf. [4]).

THEOREM 1.1. *With the above notation, there do not exist $\rho(n)$ linearly independent vector fields on S^{n-1}.*

This theorem asserts that the known positive result, stated above, is best possible. Like the theorems given below, it is copied without change of numbering from a longer paper now in preparation.

Theorem 1.1 may be deduced from the following result (cf. [1]).

THEOREM 1.2. *The truncated projective space $RP^{m+\rho(m)}/RP^{m-1}$ is not coreducible; that is, there is no map $f\colon RP^{m+\rho(m)}/RP^{m-1} \to S^m$ such that the composite*

$$S^m = RP^m/RP^{m-1} \xrightarrow{i} RP^{m+\rho(m)}/RP^{m-1} \xrightarrow{f} S^m$$

has degree 1.

Theorem 1.2 is proved by employing the "extraordinary cohomology theory" $K(X)$ of Atiyah and Hirzebruch [2; 3]. If our truncated projective space X were coreducible, then the group $K(X)$ would split as a direct sum, and this splitting would be compatible with any "cohomology operations" that one might define in the "cohomology theory" $K(X)$.

THEOREM 5.1. *It is possible to define operations*

$$\Psi_\Lambda^k\colon K_\Lambda(X) \to K_\Lambda(X)$$

for any integer k (positive, negative or zero) and for $\Lambda=R$ (real numbers) or $\Lambda=C$ (complex numbers). These operations have the following properties.

(i) *Ψ_Λ^k is natural for maps of X.*

(ii) *Ψ_Λ^k is a homomorphism of rings with unit.*

(iii) *The following diagram is commutative.*

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$$\begin{array}{ccc} K_R(X) & \xrightarrow{\Psi_R^k} & K_R(X) \\ c\downarrow & & \downarrow c \\ K_C(X) & \xrightarrow{\Psi_C^k} & K_C(X) \end{array}$$

(Here the homomorphism c is induced by "complexification" of real bundles.)

(iv) $\Psi_\Lambda^k \Psi_\Lambda^l = \Psi_\Lambda^{kl}$.

(v) Ψ_Λ^1 *and* Ψ_R^{-1} *are identity functions.* Ψ_Λ^0 *assigns to each bundle over X the trivial bundle with fibres of the same dimension.* Ψ_C^{-1} *assigns to each complex bundle over X the "complex conjugate" bundle.*

(vi) *If* $\xi \in K_C(X)$ *and* $\mathrm{ch}_q \xi$ *denotes the 2q-dimensional component of the Chern character* ch ξ, *then*

$$\mathrm{ch}^q(\Psi_C^k \xi) = k^q \, \mathrm{ch}^q \xi.$$

This theorem is proved using virtual representations of groups. By (iv), (v) it is sufficient to define Ψ_Λ^k for $k>0$. One can define polynomials Q_n^k by setting

$$(x_1)^k + (x_2)^k + \cdots + (x_n)^k = Q_n^k(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

where σ_i is the ith elementary symmetric function of $x_1, x_2, \cdots, x_n$. One can define a virtual representation of $GL(n, \Lambda)$ by setting

$$\psi_n^k = Q_n^k(E_\Lambda^1, E_\Lambda^2, \cdots, E_\Lambda^n)$$

where E_Λ^i denotes the ith exterior power representation. The operations Ψ_Λ^k are induced by the virtual representations ψ_n^k.

The values of our groups $K(X)$ and of our operations in them are given by the following result. In order to state it, we define $\phi(n, m)$ to be the number of integers s such that $m < s \leqq n$ and $s \equiv 0$, 1, 2 or 4 mod 8.

THEOREM 7.4. *Assume* $m \not\equiv -1$ mod 4. *Then* $\tilde{K}_R(RP^n/RP^m) = Z_{2^f}$, *where* $f = \phi(m, n)$. *If* $m=0$ *then the canonical real line-bundle* ξ *yields a generator* $\lambda = \xi - 1$, *and the polynomials in* λ *are given by the formula*

$$\lambda Q(\lambda) = Q(-2)\lambda,$$

where Q is any polynomial with integer coefficients. Otherwise the projection $RP^n \to RP^n/RP^m$ *maps* $\tilde{K}_R(RP^n/RP^m)$ *isomorphically onto the subgroup of* $\tilde{K}_R(RP^n)$ *generated by* λ^{g+1}, *where* $g = \phi(m, 0)$. *We write* $\lambda^{(g+1)}$ *for the element in* $\tilde{K}_R(RP^n/RP^m)$ *which maps into* λ^{g+1}.

In the case $m \equiv -1 \bmod 4$ *we have*

$$\tilde{K}_R(RP^n/RP^{4t-1}) = \tilde{K}_R(RP^n/RP^{4t}) + Z;$$

here the first summand is embedded by an induced homomorphism and the second is generated by a suitable element $\bar{\lambda}^{(g)}$, *where* $g=\phi(4t, 0)$.

The operations are given by the following formulae.

$$\text{(i)} \qquad \Psi_R^k \lambda^{(g+1)} = \begin{cases} 0 & (k \text{ even}), \\ \lambda^{(g+1)} & (k \text{ odd}); \end{cases}$$

$$\text{(ii)} \qquad \Psi_R^k \bar{\lambda}^{(g)} = k^{2t}\bar{\lambda}^{(g)} + \begin{cases} (1/2)k^{2t}\lambda^{(g+1)} & (k \text{ even}), \\ (1/2)(k^{2t}-1)\lambda^{(g+1)} & (k \text{ odd}). \end{cases}$$

This theorem is proved by deducing results in the following order:
(i) Results on complex projective spaces for $\Lambda = C$.
(ii) Results on real projective spaces for $\Lambda = C$.
(iii) Results on real projective spaces for $\Lambda = R$.

References

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