MAGNITUDE OF THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS OF NEGATIVE DIMENSION

BY JOSEPH LEHNER

Communicated by L. Bers, July 21, 1961

1. Let Γ be an H-group, i.e., Γ is a group of linear transformations of the upper half-plane $\mathfrak{H}$ on itself that is discontinuous in $\mathfrak{H}$, not discontinuous at any real point, possesses translations, and admits a fundamental region bounded by a finite number of sides. Let F be regular in $\mathfrak{H}$ and at the parabolic vertices of Γ, and

$$(1) \qquad F(V\tau) = \epsilon(V)(c\tau + d)^r F(\tau) \qquad V\,(\dot{c}\dot{d}) \in \Gamma,\ \tau \in \mathfrak{H},$$

where ϵ is a multiplier system for Γ and $-r$. Then F has a Fourier series:

$$(2) \qquad F(\tau) = \sum_{m=0}^{\infty} a_m e((m + \alpha)\tau/\lambda), \qquad \operatorname{Im} \tau > 0,$$

where $e(u) = \exp(2\pi i u)$; α and λ are defined below.

The order of magnitude of the Fourier coefficients a_m has been actively investigated for many years. Recently Petersson [2] gave estimates for forms of small negative dimension $(0<r<2)$, a range inaccessible by the usual methods. He proved:

$$(3) \qquad a_m = O(m^{r/2}), \quad 0 < r < 2, \quad r \neq 2^{-h} \quad \text{for } h = 0, 1, 2, \cdots;$$

$$(4) \qquad a_m = O(m^{r/2} \log^{r/2} m), \quad r = 2^{-h} \quad \text{for } h = 0, 1, 2, \cdots.$$

The object of this note is a slight improvement of these estimates. *We shall show that* (4) *is superfluous and that, in fact,*

$$(5) \qquad a_m = O(m^{r/2})$$

holds for all r in the range $0<r<2$.

2. We shall use our variant of the circle method (cf. [1]). Since we are interested in the Fourier coefficients (i.e., expansion coefficients at $i\infty$), it is necessary to modify the method slightly. (Also we write $-r$ for the dimension of the form, while in [1] we wrote r.) Select a fundamental region R_0 with cusp at $p_0 = i\infty$; denote the remaining inequivalent cusps in R_0 by $p_1, \cdots, p_s$. Let $S_0 = (1\ \lambda | 0\ 1)$, $\lambda > 0$, generate the subgroup of Γ fixing ∞, and let $\epsilon(S) = e(\alpha)$, $0 \leqq \alpha < 1$. Define λ_j and α_j correspondingly for $j = 1, \cdots, s$. We have the expansions, valid in $|t_j| < 1$, $|t| < 1$:

$$(A_j\tau)^{-r} t_j^{-\alpha_j} F(\tau) = f_j(t_j) = \sum_{m=0}^{\infty} a_m^{(j)} t_j^m, \qquad t_j = e(A_j\tau/\lambda_j), j > 0,$$

$$e(-\alpha\tau/\lambda)F(\tau) = f(t) = \sum_{m=0}^{\infty} a_m t^m, \qquad t = e(\tau/\lambda), j = 0,$$

with $A_j = (0\ -1 | 1\ -p_j)$, $j > 0$; $A_0 = (1\ 0 | 0\ 1)$. In terms of f_j, f, the transformation equation (1) can be written

$$(6)\ f\,(e\,(w/\lambda)) = \eta\cdot\epsilon^{-1}(V)(c_j w + d_j)^{-r} e\,(\alpha_j\, w'/\lambda_j - \alpha\, w/\lambda) f_j\,(e\,(w'/\lambda_j))$$

for $j = 0, 1, \cdots, s$, $w' = A_j V w = (a_j w + b_j)/(c_j w + d_j)$, and $|\eta| = 1$.

From (6) we now have

$$\lambda a_m = \int_L f(e(w/\lambda))e(-mw/\lambda)dw, \tag{7}$$

L being the segment: $0 \leqq x < \lambda$, $y = N^{-2}$, $N > 0$, $w = x + iy$. From [1] we take the following facts:

$$L = \dot{\bigcup_{j=-1}^{s}} \bigcup_{V \in M_j} I_j(V), \tag{8}$$

where for $j = 0, \cdots, s$, $I_j(V)$ is the interval

$$(-d_j/c_j - x_1 + iN^{-2}, -d_j/c_j + x_1 + iN^{-2}),$$

where $x_1 < 2c_j^{-1}N^{-1}h^{-1/2}$; $I_{-1}(V)$ is a finite union of intervals. (Note that the range $j = 0, \cdots, s$ corresponds to $j = 1, \cdots, s$ in [1]; $j = 0$ must be included here because $p_0 = i\infty$ is a cusp of R_0. What we denote here by I_{-1} and M_{-1} were called I_0 and M_0 in [1].) Here $h > 0$ depends only on Γ. $M_j = M_j(N)$, $j \geqq 0$, is the finite set

$$M_j = \{V \in \Gamma \mid 0 < c_j < Nh^{-1/2},\ -\kappa/N \leqq -d_j/c_j < \lambda + \kappa/N,$$

$$0 \leqq a_j/c_j < \lambda\}, \qquad \kappa = (1/c_j^2 - N^{-2})^{1/2}. \tag{9}$$

Moreover

$$\sum_{j=-1}^{s} \sum_{V \in M_j} |I_j(V)| = \lambda, \qquad |I_j| = \text{length of } I_j. \tag{10}$$

Finally, $\operatorname{Im} w' \geqq h$ for w on I_j, $j \geqq 0$, and $0 < h_0 < \operatorname{Im} w' < h_1$ for w on I_{-1}.

We are now prepared to estimate a_m from (7). Using the partition (8) we get

$$\lambda a_m = \sum_{j=0}^{s} \sum_{V\in M_j} \int_{I_j(V)} + \sum_{V\in M_{-1}} \int_{I_{-1}} \{f(e(w/\lambda))e(-mw/\lambda)dw\} = T_1 + T_2.$$

In the integrals of T_1 apply (6):

$$| T_1 | \leqq \exp(CmN^{-2}) \cdot \sum_{j=0}^{s} \sum_{V\in M_j} \int_{I_j} | c_j w + d_j |^{-r}$$
$$\cdot \exp(-C \operatorname{Im} w') \left| f_j (e(w'/\lambda_j)) \right| dx,$$

where C denotes a general constant independent of m and N. Since $\operatorname{Im} w' \geqq h > 0$, f_j is bounded on I_j, so that

$$| T_1 | \leqq C \exp(CmN^{-2}) \sum_{j=0}^{s} \sum_{V\in M_j} \int_{I_j} | c_j w + d_j |^{-r} dx.$$

We estimate trivially:

$$\int_{I_j} | c_j w + d_j |^{-r} dx = 2 \int_0^{x_1} | c_j^2 u^2 + c_j^2 N^{-4} |^{-r/2} du < 2c_j^{-r} \int_0^{N^{-2}} N^{2r} du$$
$$+ 2c_j^{-r} \int_{N^{-2}}^{c^{-1}N^{-1}} u^{-r} du = O(c^{-r} N^{2r-2}) + O(c^{-1}N^{r-1}),$$

provided $N^{-2} < x_1$; otherwise the left member is already dominated by the first integral after the inequality sign. Hence

$$| T_1 | \leqq C \exp(CmN^{-2}) \sum_{j=0}^{s} \sum_{V\in M_j} \{ c^{-r} N^{2r-2} + c^{-1} N^{r-1} \}.$$

The inner sum is, from (9), less than a sum over (c_j, d_j) with $0 < c_j < Nh^{-1/2}$, $-\alpha \leqq -d_j/c_j < \beta$, where α, β are positive constants depending on Γ and N. Hence we have

$$\sum_{V\in M_j} c^{-a} = O(N^{2-a}), \qquad 0 < a < 2$$

(cf. [2, (3.8)]), an estimate that in essence goes back to Poincaré. For each $m \geqq 1$ choose $N = m^{1/2}$ and use (10); we get $T_1 = O(m^{r/2})$.

By similar arguments we can show $T_2 = O(m^{r/2})$, and this gives the desired estimate (5) for all r in $0 < r < 2$.

3. The method can be applied also when $r \geqq 2$. It yields

$$a_m = O(m \log m), \qquad r = 2,$$
$$a_m = O(m^{r-1}), \qquad r > 2.$$

These estimates also appear in Petersson's paper.

REFERENCES

1. J. Lehner, *The Fourier coefficients of automorphic forms on horocyclic groups.* II, Michigan Math. J. vol. 6 (1959) pp. 173–193.

2. H. Petersson, *Über Betragmittelwerte und die Fourier-Koeffizienten der ganzen automorphen Formen*, Arch. Math vol. 9 (1958) pp. 176–182.

MICHIGAN STATE UNIVERSITY