THE EMBEDDING OF TWO-SPHERES IN THE FOUR-SPHERE

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1. **Introduction.** An embedding h, of S^2 in S^4 will be called *smooth* if h can be extended to an embedding of $S^2 \times E^2$ in S^4, where E^2 is the open unit disc in the plane and S^2 is identified with $S^2\times$origin. The embedding h will be called *semi-linear* if h is a simplicial homeomorphism of some rectilinear subdivision of the boundary of a 3-simplex with a subcomplex of some rectilinear subdivision of the boundary of a 5-simplex. If S^2 is semi-linearly embedded in S^4 and v is a vertex of S^2, then the link of v on S^2 is a simple closed curve, while the link of v in S^4 is a three-sphere. If this simple closed curve is unknotted in this three-sphere, then S^2 is said to be *locally flat at v*. If $S^2 \subset S^4$ is locally flat at all of its vertices, then it is said to be *locally flat.* Noguchi has shown in [1, Theorem 3] that every locally flat semi-linear S^2 in S^4 is smoothly embedded, and that its regular neighborhoods are homeomorphic to $S^2 \times E^2$.

Let S^2 have a neighborhood $S^2 \times E^2$ in S^4. Let $E^{2\prime}$ be the open disc of radius 1/2. Then $A^4 = S^4 - (S^2 \times E^{2\prime})$ is an orientable four-manifold whose single boundary component is homeomorphic to $S^2 \times S^1$. A^4 is called an *exterior* of S^2 in S^4. If the boundary component of A^4 is removed, the remaining open manifold is clearly homeomorphic to $S^4 - S^2$. I do not know whether two different exteriors of a smooth S^2 in S^4 are necessarily homeomorphic. In the locally flat semi-linear case, however, we can define the exterior to be the complement in S^4 of any open regular neighborhood of S^2. Then *the* exterior is well-defined, for one can always find a semi-linear homeomorphism of S^4 onto itself taking one regular neighborhood of S^2 onto any other (see [2]).

Two pairs (S^4, S^2) and $(S^{4\prime}, S^{2\prime})$ are said to be *equivalent* (or homeomorphic) if there is a homeomorphism h of S^4 onto $S^{4\prime}$ which carries S^2 onto $S^{2\prime}$. We can then in any case ask how strong an invariant of the pair (S^4, S^2) we get by considering the topological type of an exterior A^4 of S^2 in S^4. In other words, how many nonequivalent smooth embeddings of S^2 in S^4 can have an exterior homeomorphic to A^4?

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THEOREM 1. *There are at most two nonequivalent smooth embeddings of a two-sphere in the four-sphere with a given exterior.*

The proof will be outlined in the following sections and will appear in detail elsewhere. While it is easy to show that an exterior is a complete invariant for certain classical embeddings of S^2 in S^4, the question of whether this is always so for all smooth embeddings is still open.

Let D^2 be the closed unit disc in the plane and S^1 its boundary. Then every smooth embedding of S^2 in S^4 with an exterior homeomorphic to A^4 can be obtained, up to homeomorphism, by identifying the boundary $S^2\times S^1$ of $S^2\times D^2$ with the boundary $S^2\times S^1$ of A^4 via some homeomorphism h of $S^2\times S^1$. Furthermore, it is easily seen that the pairs obtained by using two isotopic homeomorphisms of $S^2\times S^1$ are equivalent. Thus instead of considering the group of all homeomorphism of $S^2\times S^1$, we can factor out the normal subgroup consisting of those homeomorphisms which are isotopic to the identity. Under the compact-open topology, this subgroup is the arcwise connected component of the identity. The resulting quotient group, $\mathcal{H}(S^2\times S^1)$, is called the group of *homeomorphism types* of $S^2\times S^1$. It has also been called the *homeotopy group* of $S^2\times S^1$ (see [**3**]).

It is possible that some adjunctions of $S^2\times D^2$ with A^4 may not even give S^4 for the adjunction space, while other nonisotopic identifications may nevertheless yield equivalent pairs. In any case, a natural beginning for the proof of Theorem 1 is to compute the group $\mathcal{H}(S^2\times S^1)$.

2. **The group of homeomorphism types of** $S^2\times S^1$. A homeomorphism of $S^2\times S^1$ induces an automorphism of $H_1(S^2\times S^1; Z)\approx Z$ and an automorphism of $H_2(S^2\times S^1; Z)\approx Z$, each of which depends only on the isotopy class of the homeomorphism. Thus we get a homomorphism ϕ: $\mathcal{H}(S^2\times S^1)\rightarrow Z_2\oplus Z_2$. Let r: $S^2\rightarrow S^2$ be the antipodal map and s: $S^1\rightarrow S^1$ the map induced on the unit circle by complex conjugation. Let $\mathcal{H}'\approx Z_2\oplus Z_2$ be the subgroup of $\mathcal{H}$ consisting of the isotopy classes of the maps $(1, 1)$, $(1, s)$, $(r, 1)$, (r, s). Let ρ be the isomorphism of $Z_2\oplus Z_2$ with $\mathcal{H}'\subset\mathcal{H}$ determined by the condition $\phi\rho=1$.

The main result is that the kernel of ϕ is isomorphic to Z_2. But a normal subgroup of order two is central. This and the fact that ϕ splits implies that $\mathcal{H}\approx Z_2\oplus Z_2\oplus Z_2$.

THEOREM 2. $\mathcal{H}(S^2\times S^1)\approx Z_2\oplus Z_2\oplus Z_2$.

As just remarked, it remains to be shown that ker $\phi\approx Z_2$. The proof proceeds in seven stages, during which we think of S^2 as the unit sphere in three-space and of S^1 as the space of real numbers modulo 1. Denote by **1** the identity homeomorphism of $S^2 \times S^1$. Let Φ_α denote a rotation of S^2 about a diameter through the north and south poles through an angle $2\pi\alpha$ in some fixed direction. Define the homeomorphism T of $S^2 \times S^1$ by

$$T(x, t) = (\Phi_t(x), t).$$

By the *genus* of a homeomorphism of $S^2 \times S^1$ we shall mean the image of its isotopy class under ϕ. Thus the kernel of ϕ consists of the isotopy classes of the homeomorphisms of genus (1, 1). Both **1** and T are of genus (1, 1). We show then that any homeomorphism of $S^2 \times S^1$ of genus (1, 1) is isotopic either to **1** or to T. Let C be a small circle on S^2 about the north pole. In the first five stages we deform an arbitrary homeomorphism, h, of genus (1, 1) until $h/(S^2 \times 0) \cup (C \times S^1)$ is either the identity or else coincides with $T/(S^2 \times 0) \cup (C \times S^1)$. In stage six, we deform h into either **1** or T. The final stage uses the homotopy classification of maps of $S^2 \times S^1$ into S^2 given by Pontrjagin, [**4**], to show that T is not homotopic to **1**; a fortiori, not isotopic to **1**.

COROLLARY TO THEOREM 2. *Two homeomorphisms of $S^2 \times S^1$ are isotopic if and only if they are homotopic.*

3. Two-spheres in the four-sphere with a given exterior. Theorem 2 tells us that there are at most *eight* nonequivalent smooth embeddings of S^2 in S^4 with a given exterior.

The extendibility of a homeomorphism of $S^2 \times S^1$ to a homeomorphism of $S^2 \times D^2$ depends only on the isotopy class of the homeomorphism. Thus, if the isotopy class of the homeomorphism is contained in the subgroup $\mathcal{H}'$ of $\mathcal{H}$, then the homeomorphism can be extended to a homeomorphism of $S^2 \times D^2$. We then get the following theorem, in which A^4 denotes an exterior of S^2 in S^4.

THEOREM 3. *Let the pair (M^4, S^2) be obtained by joining $S^2 \times D^2$ to A^4 by the homeomorphism f of $S^2 \times S^1$. Similarly, let the pair (N^4, S^2) be obtained by joining $S^2 \times D^2$ to A^4 by the homeomorphism g of $S^2 \times S^1$. If the isotopy class of $g^{-1}f$ is an element of $\mathcal{H}'$, then (M^4, S^2) is homeomorphic to (N^4, S^2).*

But $\mathcal{H}'$ is of index two in $\mathcal{H}$, from which Theorem 1 follows.

4. Concluding remarks. If we join the boundary of $S^2 \times D^2$ to that of A^4 by the homeomorphism T of $S^2 \times S^1$, we get a manifold M^4 and a distinguished smooth two-sphere S^2 in M^4. The original pair (S^4, S^2)

and the new pair (M^4, S^2) give the only possible smooth embeddings of a two-sphere in a four-manifold with an exterior homeomorphic to A^4. The following assertions hold:

(4.1) M^4 is a homotopy four-sphere.

(4.2) If the two-sphere $S^2 \times 0$ on the boundary of A^4 is itself the boundary of a three-sphere with handles, "nicely" situated in A^4, then the pair (M^4, S^2) is homeomorphic to the original pair (S^4, S^2).

(4.3) In particular, any two-sphere in S^4 obtained by spinning a tame knot in three-space (see [5; 6]) is uniquely determined by the topological type of its semi-linear exterior (the complement in S^4 of any open regular neighborhood of the two-sphere).

(4.4) Modulo the conjecture that two homeomorphisms of $S^n \times S^1$ are isotopic if and only if they are homotopic, Theorems 1, 2 and 3 also hold for smooth embeddings of S^n in S^{n+2}.

References

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