THE ARGUMENT OF AN ENTIRE FUNCTION

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THEOREM. *Let $f(z)$ be an entire function of order ρ, and let $\phi(r)$ denote the number of points of the circle $|z| = r$ at which $f(z)$ is real. Then*

$$\limsup_{r \to \infty} \frac{\log \phi(r)}{\log r} \geqq \rho. \tag{1}$$

PROOF. The de la Vallée Poussin means of $f(z) = \sum b_\nu z^\nu$ are

$$V_n(z) = \frac{b_0}{2} + \sum_{\nu=1}^{n} C_{2n,n+\nu}/C_{2n,n}\, b_\nu z^\nu \qquad (n = 0, 1, 2, \cdots). \tag{2}$$

It has been shown by Pólya and Schoenberg [1] that the curve $w = f(re^{i\theta})$ crosses any straight line at least as often as the curve $w = V_n(re^{i\theta})$. Taking this line to be the real axis, let $\phi(r)$, $\phi_n(r)$ respectively, denote the number of points of $|z| = r$ at which $f(z)$, $V_n(z)$ are real. Then

$$\phi(r) \geqq \phi_n(r).$$

If $N_n(r)$ is the number of zeros of $V_n(z)$ in $|z| \leqq r$, then by the argument principle, $\phi_n(r) \geqq N_n(r)$ and thus

$$\phi(r) \geqq N_n(r) \qquad (n = 0, 1, 2, \cdots).$$

Suppose that in the circle $|z| \leqq \rho_n$, $V_n(z)$ has at least p zeros. Then for $r \geqq \rho_n$

$$\phi(r) \geqq p. \tag{3}$$

We have now the theorem of Montel (see [2, p. 113]) that in the circle

$$|z| \leqq 1 + \max_{j \leqq p} |a_j/a_n|^{1/(n-p+1)}$$

the polynomial

$$a_0 + a_1 z + \cdots + a_n z^n$$

has at least p zeros. Applying this to (2), we can take

$$\rho_n \leqq 1 + \max_{j \leqq p} \left\{ C_{2n,n+j} \left| \frac{b_j}{b_n} \right| \right\}^{1/(n-p+1)}.$$

Now if $\epsilon > 0$ is given, we have for all n

$$|b_n| \leqq An^{-n/(\rho+\epsilon)} \qquad (A > 1)$$

while, for infinitely many n, $|b_n| \geqq n^{-n/(\rho-\epsilon)}$. Thus

$$\rho_n \leqq 1 + \max_{j \leqq p} \left\{ C_{2n,n+j} \frac{Aj^{-j/(\rho+\epsilon)}}{n^{-n/(\rho-\epsilon)}} \right\}^{1/(n-p+1)}$$

for infinitely many n. Hence for such n,

$$\begin{aligned} \rho_n &\leqq 1 + \left\{ AC_{2n,n} n^{n/(\rho-\epsilon)} \right\}^{1/(n-p)} \\ &\leqq 1 + \left\{ A^{1/n} 4n^{1/(\rho-\epsilon)} \right\}^{n/(n-p)}. \end{aligned}$$

Now let α be fixed, $0 < \alpha < 1$, and take $p = \alpha n$; then for infinitely many n

$$\begin{aligned} \rho_n &\leqq 1 + \left\{ A^{1/n} 4n^{1/(\rho-\epsilon)} \right\}^{1/(1-\alpha)} \\ &\leqq \left\{ Bn^{1/(\rho-\epsilon)} \right\}^{1/(1-\alpha)}. \end{aligned}$$

Hence from (3) with $p = \alpha n$,

$$\phi((Bn^{1/(\rho-\epsilon)})^{1/(1-\alpha)}) \geqq \alpha n$$

and putting $r = \left\{ Bn^{1/(\rho-\epsilon)} \right\}^{1/(1-\alpha)}$, there is a sequence of values of r tending to infinity along which

$$\phi(r) \geqq \alpha B^{-(\rho-\epsilon)} r^{(1-\alpha)(\rho-\epsilon)}$$

whence

$$\limsup_{r \to \infty} \frac{\log \phi(r)}{\log r} \geqq (1 - \alpha)(\rho - \epsilon)$$

and the result follows since $\epsilon > 0$ and $0 < \alpha < 1$ were arbitrary.

We ask: (a) can the sign of inequality hold in (1)? (b) is it true that

$$\limsup_{n \to \infty} \frac{\log n}{\log r_n} = \rho$$

where r_n is the modulus of the zero of largest modulus of (2)?

References

1. G. Pólya and I. J. Schoenberg, *Remarks on de la Vallée Poussin means and convex conformal maps of the circle*, Pacific J. Math. vol. 8 (1958) pp. 295–334.

2. M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Mathematical Surveys, no. 3, American Mathematical Society, 1949.

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