TOPOLOGICAL EQUIVALENCE OF A BANACH SPACE WITH ITS UNIT CELL

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Communicated by Mahlon M. Day, January 3, 1961

Several years ago [8] we proved that Hilbert space is homeomorphic with both its unit sphere $\{x: \|x\|=1\}$ and its unit cell $\{x: \|x\| \leqq 1\}$. Later [9] we showed that in every infinite-dimensional normed linear space, the unit sphere is homeomorphic with a (closed) hyperplane and the unit cell with a closed halfspace. It seems probable that every infinite-dimensional normed linear space is homeomorphic with both its unit sphere and its unit cell, but the question is unsettled even for Banach spaces. Corson [4] has recently proved that every $\aleph_0$-dimensional normed linear space is homeomorphic with its unit cell. In the present note, we establish the same result for a class of infinite-dimensional Banach spaces which is *believed* to include *all* such spaces. It is *proved* to include every infinite-dimensional Banach space which is reflexive, or admits an unconditional basis, or is a separable conjugate space, or is a space CM of all bounded continuous real-valued functions on a metric space M.

We employ the following tools:

(1) If E and F are Banach spaces and u is a continuous linear transformation of E onto F, then there exist a constant $m \in]0, \infty[$ and continuous mapping v of F into E such that $uvx=x$, $vrx=rvx$, and $\|vx\| \leqq m\|x\|$ for all $x \in F$ and $r \in R$ (the real number space). If G is the kernel of u and $hy=(uy, vuy-y) \in F \times G$ for each $y \in E$, then h is a homeomorphism of E onto $F \times G$. Let $\|(p, q)\| = \max(\|p\|, \|q\|)$ for all $(p, q) \in F \times G$, and let $\xi y = (\|y\|/\|hy\|)hy$ for all $y \in E$. Then ξ is a homeomorphism of E onto $F \times G$ which carries the unit cell of E onto that of $F \times G$.

(2) If S is a closed linear subspace of a Banach space E, then E is homeomorphic with the product space $(E/S) \times S$ and the unit cell of E is homeomorphic with the unit cell of this product space (with respect to any norm compatible with the product topology).

(3) In each infinite-dimensional normed linear space, the unit cell is homeomorphic with a closed halfspace.

(4) If Q is an open halfspace in an infinite-dimensional normed linear space and p is a point in the boundary of Q, then $Q \cup \{p\}$ is homeomorphic with Q.

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(5) For each $f \in L^2]0, \infty[$ and $t \in [0, 1[$, let the function $f_t \in L^2]0, \infty[$ be defined as follows: $f_t x = tf(tx)$ for $x \in]0, 1[$; $f_t x = f(x+t-1)$ for $x \in [1, \infty[$. Then with $\eta(f, t) = (f_t, t)$, the transformation η is a homeomorphism of $L^2]0, \infty[\times[0, 1[$ onto $(L^2]0, \infty[\times]0, 1[) \cup (L^2[1, \infty[\times\{0\})$.

The existence of v and m as described in (1) follows from a theorem of Bartle and Graves [**1**, p. 404] (see also Michael [**13**]). It is easily verified that h is a homeomorphism [**10**], and homogeneity of h follows from that of u and v. Thus the transformation ξ is also homogeneous. To complete the proof of (1) it suffices to observe that $(1+m)^{-1}\|y\| \leqq \|hy\| \leqq (m\|u\|+1)\|y\|$ for all $y \in E$. Proposition (2) results from applying (1) to the canonical mapping u of E onto E/S.

The result (3) appears in [**9**]. For (5), see page 29 of [**8**]. A theorem much stronger than (4) is proved on pages 12–28 of [**8**]. When the space is nonreflexive or is an (l^p) space, (4) is explicitly a corollary of (3.3) on page 27 of [**8**]. In the general case, it follows from the reasoning (though not explicitly from any statement) in [**8**]. Also, a proof of (4) is outlined in [**11**].

A normed linear space J will be called *compressible* provided the space $J\times[0, 1[$ is homeomorphic with the space $(J\times]0, 1[) \cup (W\times\{0\})$ for some closed linear subspace W of infinite deficiency in J. (We see by (5) that Hilbert space is compressible.) A space is *h-compressible* provided it is homeomorphic with some compressible normed linear space.

THEOREM. *If a Banach space B admits a continuous linear transformation onto a Banach space E which contains an h-compressible closed linear proper subspace S, then B is homeomorphic with the unit cell of B.*

PROOF. Let G denote the kernel of the continuous linear transformation of B onto E. By (1), B is homeomorphic with the product space $P = E\times G$ and the unit cell of B is homeomorphic with the unit cell U of P. To establish the theorem, it suffices to show that P is homeomorphic with U. Since S is a closed linear proper subspace of E, the subspace $T = S\times\{0\}$ must be in a closed hyperplane V in P. The unit cell U of P is homeomorphic with $V\times[0, 1[$ by (3), and V is homeomorphic with $(V/T)\times T$ by (2), so U is homeomorphic with $(V/T)\times(T\times[0, 1[)$. Clearly P itself is homeomorphic with $V\times]0, 1[$ and hence with $(V/T)\times(T\times]0, 1[)$, so to complete the proof it suffices to show that $T\times[0, 1[$ is homeomorphic with $T\times]0, 1[$. Since T is h-compressible, there exist a Banach space J homeomorphic with T and a subspace W of infinite deficiency in J such that

$J\times[0,1[$ is homeomorphic with $(J\times]0,1[)\cup(W\times\{0\})$. Let u denote the canonical mapping of J onto J/W and then let v and h be as in (1) above. Then h is a homeomorphism of J onto $(J/W)\times W$, and since $hw=(\theta, v\theta-w)$ for all $w\in W$ (where θ is the neutral element of J/W), it follows that $hW=\{\theta\}\times W$. Consequently the space $(J\times]0, 1[)\cup(W\times\{0\})$ is homeomorphic with

$$(J/W) \times W \times]0,1[\ \cup \ \{\theta\} \times W \times \{0\},$$

which in turn is homeomorphic with

$$W \times ((J/W) \times]0,1[\ \cup \ \{0\} \times \{0\}).$$

Since J/W is infinite-dimensional, it follows by (4) that the set above is homeomorphic with

$$W \times ((J/W) \times]0, 1[),$$

and hence with $J\times]0, 1[$. Reviewing the information now assembled, we see that $T\times[0, 1[$ is homeomorphic with $T\times]0, 1[$, and hence that U is homeomorphic with P. This completes the proof of the theorem.

COROLLARY. *If an infinite-dimensional Banach space B satisfies at least one of the following conditions, then B is homeomorphic with its unit cell*:

(a) *B is reflexive*;

(b) *B is a linear subspace of a Banach space which admits an unconditional basis*;

(c) *B is a norm-separable w^*-closed linear subspace of a conjugate space*;

(d) *B is the space CN of all bounded continuous real-valued functions on a normal space N which contains a closed infinite metrizable subset.*

PROOF. In view of the theorem and the fact (by (5)) that Hilbert space is compressible, it suffices in each case to produce a continuous linear transformation of B onto a Banach space E which contains a closed linear proper subspace S which is homeomorphic with Hilbert space. When B is reflexive, let $E=B$ and let S be an infinite-dimensional separable closed linear proper subspace of E. Then S is reflexive and hence (by a theorem of Kadeč [7]) homeomorphic with Hilbert space.

If B is a subspace of a space which admits an unconditional basis, a theorem of James [5] and Bessaga and Pełczyński [2] asserts that either B is reflexive or some linear subspace of B is linearly homeomorphic with the space (l) or the space (c_0). But the latter two spaces are known to be homeomorphic with Hilbert space (by results of Mazur [12] and Kadeč [6]) and the desired conclusion follows.

Now suppose B is a separable conjugate space or, more generally, that B is a norm-separable w^*-closed linear subspace of a conjugate Banach space L^*. Let $f \in B \sim \{0\}$, $x \in L$ with $fx = 1$, and $S = \{g \in E: gx = 0\}$. Then S is a w^*-closed linear proper subspace of B, and must be homeomorphic with Hilbert space by a theorem in [10]. Consequently, B is homeomorphic with its unit cell.

Finally, let B and N be as in (d). Then there is a countably infinite closed subset Z of N which consists of either a discrete set or a convergent sequence together with its limit point. For each $\phi \in CN$ let $u\phi = \phi | Z \in CZ$. Then u is a continuous linear transformation of CN onto CZ, and CZ is equivalent to either the space (m) or the space (c_0). In either case, CZ has the h-compressible space (c_0) as a closed linear proper subspace, and the desired conclusion follows upon applying the theorem.

Note that the topological equivalence of every infinite-dimensional Banach space with its unit cell would be implied by the generally expected affirmative answer to the following question: Are all infinite-dimensional separable Banach spaces homeomorphic? Recent results on this problem have been obtained by Bessaga and Pełczyński [3].

At least for reflexive spaces, the corollary above can be significantly improved. The method is that of [8, pp. 30–31] in conjunction with the above techniques and the result is as follows:

THEOREM. *Suppose E is an infinite-dimensional reflexive Banach space and C is a closed convex subset of E which has nonempty interior. Then C is homeomorphic with E and the boundary of C is homeomorphic with E or with $E \times S^n$ for some finite n and n-sphere S^n.*

The following problems seem worthy of mention: Are all infinite-dimensional separable Banach spaces h-compressible? (An affirmative answer implies that every infinite-dimensional Banach space is homeomorphic with its unit cell.) Are all infinite-dimensional Banach spaces compressible? Are $\aleph_0$-dimensional normed linear spaces compressible? Note that for Hilbert space, the compressibility was achieved by means of a continuous family of affine homeomorphisms. How generally is this possible?

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