DERIVATIONS OF COMMUTATIVE BANACH ALGEBRAS

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In [2] Singer and Wermer showed that a bounded derivation in a commutative Banach algebra $\mathfrak{A}$ necessarily maps $\mathfrak{A}$ into the radical $\mathfrak{R}$. They conjectured at this time that the assumption of boundedness could be dropped. It is a corollary of results proved below that if $\mathfrak{A}$ is in addition regular and semi-simple, this is indeed the case.

What is actually proved here is that under the above hypotheses, if D is a derivation of $\mathfrak{A}$ into $C(\Phi_{\mathfrak{A}})$,[2] $\Phi_{\mathfrak{A}}$ the structure space of $\mathfrak{A}$, then D is a bounded operator from $\mathfrak{A}$ to $C(\Phi_{\mathfrak{A}})$. The topologies are the norm topology in $\mathfrak{A}$ and the sup norm topology in $C(\Phi_{\mathfrak{A}})$. An application of the closed graph theorem shows that if D maps $\mathfrak{A}$ into itself, D must be a bounded operator in $\mathfrak{A}$, hence by the Singer, Wermer theorem, $D=0$.

If $\mathfrak{A}$ is regular but not semi-simple, then it follows from the above that D will map $\mathfrak{A}$ into $\mathfrak{R}$ provided that D maps $\mathfrak{R}$ into $\mathfrak{R}$. This the author can verify only if $\mathfrak{R}$ is nilpotent.

In what follows $\mathfrak{A}$ will always denote a regular, commutative, semi-simple Banach algebra with norm $\|\cdot\|$. Applying the Gelfand isomorphism we will identify $\mathfrak{A}$ and the corresponding subalgebra of $C(\Phi_{\mathfrak{A}})$. For convenience we also will assume $\mathfrak{A}$ possesses an identity. It is easily seen that this doesn't affect the generality of the results.

Let $\mathfrak{M}_\phi$ be a maximal ideal of $\mathfrak{A}$, and ϕ the corresponding point in $\Phi_{\mathfrak{A}}$. It is noted in [2] that there exists a derivation D of $\mathfrak{A}$ into some semi-simple extension $\mathfrak{B}$ of $\mathfrak{A}$ iff $\mathfrak{M}_\phi^2 \neq \mathfrak{M}_\phi$ for some maximal ideal $\mathfrak{M}_\phi$. In fact $\mathfrak{B}$ may be taken to be $B(\Phi_{\mathfrak{A}})$, the ring of bounded complex functions on $\Phi_{\mathfrak{A}}$. For if this condition is satisfied, following Singer and Wermer, we define by Zorn's Lemma a nontrivial linear functional f_ϕ on $\mathfrak{A}$ which annihilates $\mathfrak{M}_\phi^2$ and the identity. If we define D by

$$\begin{aligned} Dx(\phi') &= 0, \qquad \phi' \in \Phi_{\mathfrak{A}}, \qquad \phi' \neq \phi, \qquad x \in \mathfrak{A}, \\ Dx(\phi) &= f_\phi(x), \end{aligned}$$

it is easily seen that D is a derivation of $\mathfrak{A}$ into $B(\Phi_{\mathfrak{A}})$. D is in general unbounded, but if $\overline{\mathfrak{M}_\phi^2} \neq \mathfrak{M}_\phi$, f_ϕ, and consequently D, may be chosen (via the Hahn-Banach Theorem) to be bounded. Modifying the

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[2] $C(\Phi_{\mathfrak{A}})$ denotes the algebra of continuous complex functions on the space $\Phi_{\mathfrak{A}}$.

terminology of Singer and Wermer somewhat we refer to both the functionals f_ϕ and the associated operators D as point derivations. The main result of this note is that any derivation D of $\mathfrak{A}$ into $B(\Phi_{\mathfrak{A}})$ is the sum of a bounded derivation and finitely many unbounded point derivations.

The key to the argument is the following result from [1, §3] stated in a form suitable to our needs.

THEOREM 1. *Let* $\|\cdot\|_1$ *be a norm on* $\mathfrak{A}$ *under which* $\mathfrak{A}$ *is a normed algebra. Let* $\mathcal{G}$ *be the class of open sets* G *for which there exist constants* M_G *satisfying*

$$\|x\|_1 \leqq M_G\|x\|, \qquad x \in \mathfrak{A}; \qquad c(x) \subset G.^3$$

Then there exists a finite subset F *of* $\Phi_{\mathfrak{A}}$, *called the singularity set of the norm* $\|\cdot\|_1$, *with the following two properties*:

(1) *If* G *is open and* $\overline{G}\cap F=\emptyset$, *then* $G\in\mathcal{G}$.
(2) *If* $G\in\mathcal{G}$, *then* $G\cap F=\emptyset$.

We now state and prove the result of the note.

THEOREM 2. *Let* D *be a derivation of* $\mathfrak{A}$ *into* $B(\Phi_{\mathfrak{A}})$. *Then there exists a finite subset* F *of* $\Phi_{\mathfrak{A}}$ *and a bounded derivation* D_1 *of* $\mathfrak{A}$ *into* $B(\Phi_{\mathfrak{A}})$ *such that if* $D_2=D-D_1$, *then* $D_2x(\phi)=0$, $x\in\mathfrak{A}$ *and* $\phi\in\Phi_{\mathfrak{A}}-F$. *For* $\phi\in F$, $f_\phi(x)\equiv D_2x(\phi)$ *is an unbounded point derivation. If for each* $x\in\mathfrak{A}$, $Dx\in C(\Phi_{\mathfrak{A}})$, *then* $F=\emptyset$ *and* D *is a bounded operator.*

PROOF. Re-norm $\mathfrak{A}$ by defining for $x\in\mathfrak{A}$ $\|x\|_1=\|x\|+\|Dx\|_\infty$ where $\|y\|_\infty=\sup_{\phi\in\Phi_{\mathfrak{A}}}|y(\phi)|$. Clearly $\mathfrak{A}$ is a normed algebra under $\|\cdot\|_1$. Therefore if F is the singularity set for $\|\cdot\|_1$, we assert $f_\phi(x)\equiv Dx(\phi)$ is a bounded linear functional on $\mathfrak{A}$ iff $\phi\notin F$. If $\phi\notin F$, then by the regularity of $\mathfrak{A}$ there exists $h_\phi\in\mathfrak{A}$ and a neighborhood V of F such that $h_\phi(\phi)=1$, $h_\phi(V)=0$. Let $\mathfrak{J}_V=\{x\in\mathfrak{A}: x(V)=0\}$. Choose an open set W, $\overline{W}\cap F=\emptyset$ such that if $x\in\mathfrak{J}_V$, then $c(x)\subset W$. Then by Theorem 1, D is bounded on $\mathfrak{J}_V$. Hence if $\{x_n\}$ is any sequence in $\mathfrak{A}$ tending to zero, then $x_nh_\phi\in\mathfrak{J}_V$ and $x_nh_\phi\to 0$. Consequently $D(x_nh_\phi)\to 0$. But $D(x_nh_\phi)(\phi)=Dx_n(\phi)+x_n(\phi)\cdot Dh_\phi(\phi)$. Therefore $f_\phi(x_n)\equiv Dx_n(\phi)\to 0$. For the converse let $H=\{\phi: f_\phi$ is bounded on $\mathfrak{A}\}$. Since $\|Dx\|_\infty<\infty$ for each $x\in\mathfrak{A}$, there exists by the principle of uniform boundedness, a constant M such that $\sup_{\phi\in H}|Dx(\phi)|\leqq M\|x\|$. If $\phi_0\in H\cap F$, pick an open set $G\subset H$, $\phi_0\in G$ and an element $y\in\mathfrak{A}$ for which $y(G)=1$ and $y(\Phi_{\mathfrak{A}}-H)=0$. Then if $x\in\mathfrak{A}$ and $c(x)\subset G$, we have $xy=x$. Therefore $Dx=yDx+xDy$ and

[3] $c(x)$ denotes the carrier of the function x.

$$\|Dx\|_\infty \leqq \sup_{\phi\in H} |y(\phi)\cdot Dx(\phi)| + \|x\|\cdot\|Dy\|_\infty$$
$$\leqq \{\|y\|_\infty\cdot M + \|Dy\|_\infty\}\|x\|.$$

This contradicts property (2) of Theorem 1.

If $F\neq\emptyset$ and D is unbounded, we define D_1 by

$$D_1x(\phi) = Dx(\phi), \qquad \phi\notin F,$$
$$= 0, \qquad \phi\in F.$$

Again applying the uniform boundedness principle it follows that D_1 is a bounded operator from $\mathfrak{A}$ to $B(\Phi_{\mathfrak{A}})$. The statement about D_2 is clear.

To complete the proof we observe first that if ϕ is isolated in $\Phi_{\mathfrak{A}}$, then $\phi\notin F$. In fact for such ϕ, $Dx(\phi)=0$. For let k_ϕ be the characteristic function of $\{\phi\}$. Then $k_\phi\in\mathfrak{A}$ and for $x\in\mathfrak{A}$ $D(k_\phi x)(\phi)=0$. Hence $Dx(\phi)=-x(\phi)\cdot Dk_\phi(\phi)=0$. Consequently $\overline{\Phi_{\mathfrak{A}}-F}=\Phi_{\mathfrak{A}}$. Therefore if for each $x\in\mathfrak{A}$, Dx is a continuous function on $\Phi_{\mathfrak{A}}$, it follows that $\|Dx\|_\infty=\sup_{\phi\in\Phi_{\mathfrak{A}}-F}|Dx(\phi)|\leqq M\|x\|$. This completes the proof.

COROLLARY. *Let $\mathfrak{B}$ be a subalgebra of $C(\Phi_{\mathfrak{A}})$ containing $\mathfrak{A}$. If $\mathfrak{B}$ is a Banach algebra under some norm and D is a derivation of $\mathfrak{A}$ into $\mathfrak{B}$, then D is a bounded operator. If D maps $\mathfrak{A}$ into itself, then $D\equiv 0$.*

PROOF. The first result follows by the closed graph theorem. An application of the theorem of Singer and Wermer [2] then yields the second.

If now $\mathfrak{A}$ is not semi-simple and D maps $\mathfrak{A}$ into itself, then one may factor out the radical and apply the above corollary to prove that D maps $\mathfrak{A}$ into $\mathfrak{R}$ provided that D maps $\mathfrak{R}$ into $\mathfrak{R}$. If $\mathfrak{R}$ is nilpotent, this follows. For if $x^n=0$, then $0=D^nx^n=n!(Dx)^n+$terms each of which involves a positive power of x, hence belongs to the radical. Therefore $(Dx)^n\in R$, and consequently $Dx\in R$.

The validity of this result for non-nilpotent radicals is unknown to the author. Without some topological assumptions the result is of course false. Ordinary differentiation in the ring of formal power series is a derivation which does not map the radical into itself.

BIBLIOGRAPHY

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