THE MANIFOLD SMOOTHING PROBLEM[1]

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The Schoenflies Theorem in n dimensions has been proved by both Marston Morse [4] and Morton Brown [1] subject to the shell hypothesis [4]. Morse's proof leads to C^m-diffeomorphisms. We now prove the following Schoenflies Theorem for polyhedra without the shell hypothesis.

THEOREM 1.[2] *Let P^{n-1} be a combinatorial $(n-1)$-sphere in a euclidean n-space E^n, and let N be an arbitrary neighborhood of P^{n-1}. Then E^n can be mapped onto itself by a homeomorphism h which is a C^∞-diffeomorphism on E^n-N and which maps P^{n-1} onto a euclidean $(n-1)$-sphere S^{n-1}.*

The proof commences with a modification of a procedure due to H. Noguchi [5] yielding an ϵ-isotopy of E^n carrying P^{n-1}, on D^n, into a polyhedron Q^{n-1}, admitting a transverse vector field. A neighborhood of Q^{n-1} is fibred by C^∞—$(n-1)$-spheres, which permits a completion of the proof with the aid of Morse's methods [4]. His exceptional interior point can be relegated to N. The proof is inductive, requiring a partial assumption of Theorem 1 in the next lower dimension.

COROLLARY. *Given a $\delta>0$, E^n admits a δ-isotopy h_t $(0\leqq t\leqq 1)$ such that* (1) *h_t is the identity on the unbounded component of E^n-N,* (2) *$h_t(P^{n-1})\subset D^n$ $(t>0)$ and* (3) *$h_t(P^{n-1})$ is a C^∞-$(n-1)$-sphere $(t>\delta)$.*

We will call a combinatorial n-manifold *smoothable* or *nonsmoothable* according as it is or is not compatible with a differentiable structure. The known nonsmoothable manifolds include a K^8 due to Milnor [3] and a K^{10} due to Kervaire [2]. The latter is *strongly nonsmoothable,* in the sense that the topological manifold it covers, $M^{10}=|K^{10}|$, can not carry a differentiable structure, either compatible or incompatible with K^{10}.

A piecewise differentiable imbedding of a K^m in a differentiable n-manifold M^n means a homeomorphism $h: K^m\to M^n$, where h is differentiable of maximal rank on each closed simplex of K^m.

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[2] A sharpening of this theorem appears in Proc. Nat. Acad. Sci. U.S.A. vol. 47, (1961) pp. 328–330.

THEOREM 2. *A combinatorial n-manifold K^n without boundary is smoothable if and only if K^n admits a piecewise differentiable imbedding h into a differentiable M^{n+1}.*

The necessity of the condition is easy to prove. The sufficiency proof commences with an $h: K^n \to M^{n+1}$ restricted, as in the proof of Theorem 1, so that $h(K^n)$ admits a transverse vector field on M^{n+1}. Let M^{n+1} be represented as a differentiable submanifold of an E^{n+r}. With the aid of a potential function, equipotential $(n+r-1)$-manifolds surrounding $h(K^n)$ in E^{n+r} can be defined [6]. If $h(K^n)$ is two-sided in M^{n+1}, the intersection $V^{n+r-1} \cap M^{n+1}$ with M^{n+1} of an equipotential sufficiently near $h(K^n)$ falls into two components, V_1^n and V_2^n, each of which is differentiable and homeomorphic to K^n. If $h(K^n)$ is one-sided in M^{n+1}, points can be so identified in pairs on $V^{n+r-1} \cap M^{n+1}$ as to obtain a differentiable homeomorph of $h(K^n)$.

COROLLARY. *The K^8 of Milnor and K^{10} of Kervaire do not admit piecewise differentiable imbeddings in differentiable 9-manifolds and 11-manifolds respectively.*

THEOREM 3. *If there exists a nonsmoothable K^m without boundary, then there is a nonsmoothable K^n without boundary for each $n > m$.*

In particular, $K^m \times S^1$ where S^1 is a circle, is nonsmoothable, for its smoothability would imply that of K^m, by Theorem 2. Thus, all the manifolds $K^8 \times S^1 \times \cdots \times S^1$ and $K^{10} \times S^1 \times \cdots \times S^1$ are nonsmoothable, for Milnor's K^8 and Kervaire's K^{10}.

The invariants used by Milnor and Kervaire are thus freed from the dimensions for which they were defined. They are imbeddability as well as smoothability criteria.

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