INTEGRATION WITH RESPECT TO OPERATOR-VALUED FUNCTIONS

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1. Introduction. Let J be a compact subinterval of the real line. N. Wiener [7] has introduced the Banach algebra $W_p(J)$ of all complex-valued functions f such that $V_p(f) \neq \infty$, where

$$V_p(f) = \sup\left(\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p},$$

the supremum being taken over all finite partitions of J (see §7). We shall construct a family of continuous homomorphisms of the Banach algebra $W_p(J)$; this connects with the theory of multipliers of Fourier series (see §4). Our basic problem is to integrate (in the uniform operator-topology) with respect to functions that are not of bounded variation.

Given a fixed measure space $(a, \mathfrak{A}, \mu)$, let $\mathfrak{E}_r$ denote the Banach algebra of all continuous endomorphisms of $L_r(a, \mathfrak{A}, \mu)$; the relation $1 < r < \infty$ is implied throughout. Let E_r be a function on J which assumes its values in $\mathfrak{E}_r$, and let f belong to the class $D(J)$ of all simply-discontinuous,[2] complex-valued functions. The following expression

$$(1) \qquad (\mathfrak{E}_r) \int f(\lambda) \cdot dE_r(\lambda)$$

will denote what T. H. Hildebrandt [1, p. 273] calls the "*modified Stieltjes integral*"; it is the limit of a certain net of Stieltjes sums (this net is directed as in the Pollard-Moore integral [1, p. 269]). The word "limit" here implies convergence in the norm-topology of $\mathfrak{E}_r$. It is not hard to show that the integral (1) converges when E_r is of bounded variation[3]; this situation is most familiar in the case $r = 2$, when E_r is a resolution of the identity in the Hilbert space $L_2(a, \mathfrak{A}, \mu)$. Henceforth, we will allow the possibility that E_r not be of bounded variation (this possibility becomes a fact in Theorem D below).

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[2] That is, having on J at most discontinuities of the first kind.

[3] In the sense of Hille-Phillips [2, p. 59]. Bounded variation is a less restrictive condition than the bounded semi-variation hypothesis required in certain integration theories (e.g., Bartle's article in the Studia Math. vol. 15 (1956) pp. 337–352).

2. **Motivation.** Suppose that $r \neq 2$. Some integrators E_r have the following property: there exists no spectral measure M such that

$$\int \lambda \cdot M(d\lambda) = (\mathfrak{E}_r) \int \lambda \cdot dE_r(\lambda),$$

although the integral on the right-hand side converges.

3. **An operational calculus.** Let $L^0(a, \mathfrak{a}, \mu)$ be the class of all simple functions. If $T \in \mathfrak{E}_2$ we write

(2) $$|T|_r = \sup\{\|Tx\|_r : x \in L^0(a, \mathfrak{a}, \mu) \text{ and } \|x\|_r \leqq 1\};$$

it is clear that the eventuality $|T|_r \neq \infty$ implies the existence of the continuous extension (denoted T_r) of T from $L^0(a, \mathfrak{a}, \mu)$ to $L_r(a, \mathfrak{a}, \mu)$.

Suppose that E is a resolution of the identity in $L_2(a, \mathfrak{a}, \mu)$ such that

(v) $$\infty \neq \sup_{\lambda \in J} |E(\lambda)|_s \quad \textit{whenever} \quad 1 < s < \infty.$$

If $\lambda \in J$, then $E(\lambda) \in \mathfrak{E}_2$ and $|E(\lambda)|_r \neq \infty$, whence $E(\lambda)_r \in \mathfrak{E}_r$ (here $E(\lambda)_r$ again denotes the extension of $E(\lambda)$ from $L^0(a, \mathfrak{a}, \mu)$ to $L_r(a, \mathfrak{a}, \mu)$). Accordingly, we may define on J a function E_r by means of the relation $E_r(\lambda) = E(\lambda)_r$. Finally, let $I(p)$ denote the open interval with endpoints $2p/(p \pm 1)$. Under these circumstances, it can be proved that:

if $1 \leqq p < \infty$ and $r \in I(p)$, then each integral of the family

$$\left\{(\mathfrak{E}_r) \int f(\lambda) \cdot dE_r(\lambda) : f \in W_p(J)\right\}$$

converges in the norm-topology of $\mathfrak{E}_r$.

It is notable that, if $\infty > p > q > 1$, then

$$\{2\} \subset I(p) \subset I(q) \subset I(1) = \{\lambda : 1 < \lambda < \infty\},$$
$$D(J) \supset W_p(J) \supset W_q(J) \supset W_1(J) = \{\text{bounded variation}\}.$$

In other words: as the range of r expands from the Hilbert-space case $\{2\}$ to comprise the whole interval $(1, \infty)$, then $W_p(J)$ contracts into the class $W_1(J)$ consisting of all functions of bounded variation.

There exists a well-known bijection $\{T \to E^T\}$ of the class $\mathfrak{T}$ (of all self-adjoint members of $\mathfrak{E}_2$) into the class of all resolutions of the identity [6, p. 174 and p. 176]. Suppose $T \in \mathfrak{T}$, and let J be an interval that contains the spectrum of T. It will be convenient to write

$$f(T_r) = (\mathfrak{E}_r) \int f(\lambda) \cdot dE_r^T(\lambda).$$

An application of the Spectral Theorem shows easily that:

if $f(\lambda) = \sum_n \alpha_n \cdot \lambda^n$ *is a polynomial, then* $f(T_r) = \sum_n \alpha_n \cdot (T_r)^n$.

THEOREM A. *Suppose that* $T \in \mathfrak{T}$ *and let condition* (v) *be satisfied when* E *is replaced by* E^T. *If* $1 \leqq p < \infty$ *and* $r \in I(p)$, *then the mapping* $\{f \to f(T)_r\}$ *is a continuous homomorphism of the Banach algebra* $W_p(J)$ *into* $\mathfrak{E}_r$.

4. Two applications to the theory of multiplier transformations. Consider a complete orthonormal system $\{\Phi_n : n \in a\}$; accordingly, a is denumerable. In this paragraph, $\mathfrak{a}$ consists of all subsets of a, while μ is taken to be counting-measure; thus $L_r(a, \mathfrak{a}, \mu)$ becomes the sequence space usually denoted l_r, and $L^0(a, \mathfrak{a}, \mu)$ is now the class l^0 of all sequences that vanish off finite subsets of a. If $x \in l^0$, then $f_\#(x)$ will denote the sequence of Fourier coefficients of the function $f \cdot x^\frown$, where

$$(f \cdot x^\frown)(\lambda) = f(\lambda) \cdot \sum_{n \in a} x_n \cdot \Phi_n(\lambda) \qquad (\lambda \in J).$$

Let $f_\#$ be the mapping $\{x \to f_\#(x)\}$ defined on l^0. Hirschman [3] calls $f_\#$ a *"multiplier transformation."* An important problem in the theory of multiplier transformations is to find conditions on f which will insure that $|f_\#|_r \neq \infty$.

First application. Let $\{\Phi_n : n \in a\}$ be the system of normalized Legendre polynomials on $J = [-1, 1]$, and denote by T the member Δ of $\mathfrak{T}$ that is defined in [4, (2)]. The article [4] shows that condition (v) is satisfied when E is replaced by E^T. Suppose $1 \leqq p < \infty$ throughout. Our theory shows that

(i) *if* $r \in I(p)$ *and* $f \in W_p(J)$, *then* $|f_\#|_r \neq \infty$;

consequently, $f_\#$ has a continuous extension $f_{\#r}$ from l^0 to l_r. In fact, we can prove

THEOREM B. *If* $r \in I(p)$ *and* $f \in W_p(J)$, *then* $f_{\#r} = f(T_r) \in \mathfrak{E}_r$.

Second application. From now on, $\{\Phi_n : n \in a\}$ will be the trigonometric system; a is now the integer group and $J = [0, 1]$. Originally proved by Stečkin in the case $p = 1$, property (i) was discovered by Hirschman [3]; his proof is based on Stečkin's result. Theorem B is also valid in the present context, the operator T being now the Hilbert transformation defined for all x in l_2 by the relation

$$(Tx)_n = \sum_{k \in a} x_k \cdot \frac{i}{2\pi(n-k)} \qquad (k \neq n)$$

for each n in a. On the strength of Theorem A, we can prove Theorem B directly from the following well-known property: $|T|_s \neq \infty$ whenever $1 < s < \infty$.

THEOREM D. *Let T_r be the unitary shift operator defined (for all x in l_r) by the relation $T_r x = \{n \to x_{n+1}\}$; there exists a function E_r on J to $\mathfrak{E}_r$ such that*

$$T_r = (\mathfrak{E}_r) \int e^{-2\pi i \lambda} \cdot dE_r(\lambda),$$

although E_r is not of bounded variation.[3]

5. Hölder-type inequalities and the variation-norm. We now return to the general setting of §3; once again, $(a, \mathfrak{A}, \mu)$ is an arbitrary measure space, and the integrator E is a resolution of the identity satisfying (v). If $x, y \in L^0(a, \mathfrak{A}, \mu)$, then the relation

$$E_{x,y}(\lambda) = \int_a y \cdot E(\lambda) x \cdot d\mu \qquad (\lambda \in J)$$

defines a complex-valued function $E_{x,y}$. The variation-norm is defined as follows:

$$V_q(E)_r = \sup\{V_q(E_{x,y}) : x \in U_r \text{ and } y \in U_{r'}\},$$

where $U_s = \{z \in L^0(a, \mathfrak{A}, \mu) : \|z\|_s \leq 1\}$ and $r' = r/(r-1)$. When $f \in D(J)$ and $r = 2$ it is easy to verify the familiar inequality

$$\text{(ii)} \qquad \left| (\mathfrak{E}_r) \int f(\lambda) \cdot dE_r(\lambda) \right|_r \leq V_1(E)_r \cdot \|f\|_\infty,$$

where $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in J\}$.

Suppose $1 < p < \infty$ and $r \in I(p)$. Our approach involves establishing the existence of a number $q > 1$ such that $q^{-1} + p^{-1} > 1$ and

$$\text{(ii*)} \qquad \left| (\mathfrak{E}_r) \int f(\lambda) \cdot dE_r(\lambda) \right| \leq c_{r,p} \cdot V_q(E)_r \cdot (\|f\|_\infty + V_p(f)) < \infty$$

(where $c_{r,p}$ is independent of f and E), for each f in $W_p(J)$. This is closely related to a theorem of Love and L. C. Young [5]; in fact, their work is based on the same inequality[4] that we use to prove (ii*).

6. A convexity theorem for the variation-norm. With (3) as a starting-point, it is easy to define an extension F of the function

[4] Due to L. C. Young [8]. The articles by Love and Young involve only scalar-valued functions.

$\{(q, r)\to V_q(E)_r\}$ such that $\log F(\alpha^{-1}, \beta^{-1})$ is a convex function of (α, β) in the rectangle $0\leqq\alpha, \beta\leqq 1$. This fact plays a basic role in proving the results that have been presented.

7. **Remarks.** For more details concerning the Banach space $W_p(J)$, see [**5**]. If $p=1$, then $V_p(f)$ is the total variation of f. The class $W_p(J)$ becomes a Banach algebra under pointwise multiplication (and under the norm $\{f\to\|f\|_\infty+V_p(f)\}$). The article [**5**] deals with continuous linear functionals on $W_p(J)$.

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