HOLOMORPHIC DIFFERENTIALS AS FUNCTIONS OF MODULI[1]

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The purpose of this note is to strengthen the results of [3] and to indicate a very brief derivation of some theorems announced without proof in [1; 3].

We begin by indicating a correction to [3]. Let S_1 and S_2 be Riemann surfaces, f an orientation preserving (orientation reversing) homeomorphism of bounded eccentricity of S_1 onto S_2 and $[f]$ the homotopy class of f; then $(S_1, [f], S_2)$ is called an even (odd) coupled pair of Riemann surfaces. The definition of equivalence of such pairs given in [3] is imprecise and garbled by misprints. The correct definition reads: $(S_1, [f], S_2)$ and $(S_1', [f'], S_2')$ are called *equivalent* if there exist conformal homeomorphisms h_1 and h_2 with $h_1(S_1) = S_1'$, $h_2(S_2) = S_2'$ and $[h_2 f] = [f' h_1]$; the two pairs are called *strongly equivalent* if $S_2' = S_2$ and there exists a conformal homeomorphism h with $h(S_1) = S_1'$ and $[f] = [f'h]$. If S_0 is a Riemann surface, then the Teichmüller space $T(S_0)$ can be thought of as the set of strong equivalence of even pairs $(S, [f], S_0)$ (and not of simple equivalence classes as stated in [3]).[2]

From now on we assume that S_0 is a fixed closed Riemann surface of genus $g > 1$, and we write T instead of $T(S_0)$. T has a natural complex analytic structure and can be represented as a bounded domain, homeomorphic to a ball, in complex number space with coordinates (moduli) $\tau_1, \cdots, \tau_{3g-3}$ (cf. [1; 2]). Points of T will be denoted by τ. We may assume that S_0 is given as the unit disc modulo a fixed-point-free Fuchsian group, and that $\tau = 0$ corresponds to the pair $(S_0, [\text{identity}], S_0)$.

THEOREM I. *One can associate to every $\tau \in T$ a bounded Jordan domain $D(\tau)$ and $2g$ Möbius transformations $z \to A_j(z, \tau)$, $z \to B_j(z, \tau)$, $j = 1, \cdots, g$, such that the following conditions are satisfied.*

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[2] We also note the following errata to [2; 3]. On p. 94, l. 19, replace (ζ) by $\mu(\zeta)$. On p. 96, l. 15, replace the subscript j by $2j$. On p. 97, l. 21, replace C_r by C^r. On p. 100, l. 4, replace 'covering' by 'covering space.' On p. 103, equation (9) replace the exponent $3g - 3n + n$ by $3g - 3 + n$.

(i) *The boundary curve of* $D(\tau)$ *admits the parametric representation* $z=\sigma(e^{i\theta}, \tau)$, $0\leqq\theta\leqq 2\pi$, *depending holomorphically on* τ.

(ii) *The* A_j *and* B_j *depend holomorphically on* τ *and satisfy the relation*

$$\prod_{j=1}^{g} A_jB_jA_j^{-1}B_j^{-1} = 1. \tag{1}$$

For every fixed $\tau\in T$ *they generate, with the single defining relation* (1), *a fixed-point-free discrete group* $G(\tau)$ *of conformal self-mappings of* $D(\tau)$, *so that* $S(\tau)=D(\tau)/G(\tau)$ *is a closed Riemann surface of genus* g. $S(0)$ *is the surface* S_0.

(iii) *Denote by* $\alpha(\tau)$ *the basis of the fundamental group of* $S(\tau)$ *defined by* $A_1, \cdots, B_g$, *and by* f_τ *a quasiconformal mapping of* $S(\tau)$ *onto* $S(0)$ *which takes* $\alpha(\tau)$ *into* $\alpha(0)$. *Then the point* τ *corresponds to the pair* $(S(\tau), [f_\tau], S_0)$.

This statement differs from Theorem 2 in [3] primarily by the boundedness condition for $D(\tau)$ and can be obtained from that theorem without much difficulty.

We denote by M the domain in complex number space of $3g-2$ dimensions which consists of points (z, τ) with $z\in D(\tau)$ and $\tau\in T$. By Theorem 3 in [3] M is holomorphically equivalent to $T(S_0-\{p\})$ for a fixed $p\in S_0$.

We denote by $W_q(\tau)$ the (complex) vector space of holomorphic functions $\phi(z)$, $z\in D(\tau)$, for which $\phi(z)dz^q$ is invariant under $G(\tau)$; this is the same as the space of q-dimensional holomorphic differentials on $S(\tau)$, so that dim $W_q(\tau)=0$, 1, g, or $(2q-1)(g-1)$ according to whether $q<0$, $q=0$, $q=1$, or $q>1$. In $W_1(\tau)$ there exist g distinguished elements, $p_k(z, \tau)$, determined by the conditions

$$\int_z^{A_i(z,\tau)} p_k(z', \tau)dz' = \delta_{ik}; \tag{2}$$

these correspond to the normalized Abelian differentials of the first kind on $S(\tau)$ belonging to the "canonical" homology basis $a(\tau)$ determined by $\alpha(\tau)$. The period matrix of $S(\tau)$ belonging to $a(\tau)$ will be denoted by $Z(\tau)$. It has the elements

$$Z_{ik}(\tau) = \int_z^{B_i(z,\tau)} p_k(z', \tau)dz'$$

and is a point in the Siegel space of symmetric matrices with positive definite imaginary part.

We denote by $\boldsymbol{W}_q$ the vector space of holomorphic functions $\Phi(z, \tau)$, $(z, \tau) \in M$, which belong to $W_q(\tau)$ for every fixed $\tau \in T$.

THEOREM II. *Every element of* $W_q(\tau)$ *is a restriction of an element of* $\boldsymbol{W}_q$.

PROOF. Assume that $q \geqq 2$. Let C_j, $j = 1, 2, \cdots$, be a complete system of nonequivalent (with respect to (1)) words in the letters $A_1, \cdots, B_g$. If $P(t)$ is a polynomial, then the Poincaré series

$$\sum_{j=1}^{\infty} P(C_j(z, \tau))(\partial C_j(z, \tau)/\partial z)^q \tag{3}$$

converges normally in M and its sum belongs to $\boldsymbol{W}_q$. On the other hand, since $D(\tau)$ is a bounded Jordan domain and $G(\tau)$ has a compact fundamental region, Theorem 4 in [4] implies that, for a fixed τ, every element of $W_q(\tau)$ is of the form (3).

For $q = 1$ we shall show that every p_j belongs to $\boldsymbol{W}_1$ (i.e., that the normalized Abelian differentials are holomorphic functions of the moduli).

THEOREM III. *The functions* $p_k(z, \tau)$, $k = 1, \cdots, g$, *are holomorphic in* M.

PROOF. It suffices to consider p_1. We shall show that in a neighborhood of a fixed but arbitrary point $\tau_0 \in T$ we have an identity of the form

$$p_1(z, \tau) = \Phi(z, \tau)^{-1} \sum_{j=1}^{5g-5} c_j(\tau) \Phi_j(z, \tau) \tag{4}$$

where the c_j are holomorphic, $\Phi \in \boldsymbol{W}_2$, and the Φ_j are elements of $\boldsymbol{W}_3$. We first choose Φ so that $\Phi(z, \tau_0)$ vanishes at $4g-4$ points z_i which are not equivalent under $G(\tau_0)$. This is possible since the "general" holomorphic quadratic differential on $S(\tau)$ has only simple zeros (Bertini) and hence exactly $4g-4$ of those. There exist $4g-4$ holomorphic functions $\zeta_i(\tau)$ defined near τ_0, such that $\zeta_i(\tau_0) = 0$ and $\Phi(z_i + \zeta_i(\tau), \tau) = 0$. In order that the right hand side of (4) belong to $W_1(\tau)$ it is necessary and sufficient that

$$\sum_{j=1}^{5g-5} c_j(\tau) \Phi_j(z_i + \zeta_i(\tau), \tau) = 0, \qquad i = 1, \cdots, 4g-4,$$

and one sees at once that any $4g-5$ of these equations imply the $(4g-4)$th. In order that (4) hold near τ_0 the c_j must satisfy g additional linear equations which are obtained from (1) by setting $k = 1$

and choosing a fixed point z and fixed paths of integration, avoiding the points z_i. The resulting linear system, with holomorphic coefficients, for the unknown functions c_j, is uniquely solvable at τ_0 if the functions $\Phi_1, \cdots, \Phi_{6g-6}$ are chosen so as to be linearly independent for $\tau=\tau_0$. In this case the equations are also uniquely solvable for τ close to τ_0, and the solutions depend holomorphically on τ.

We proceed to derive some consequences from Theorems II and III.

(a) *The functions*

$$f_{ij} = p_i/p_j, \qquad f_{ijk} = f_k^{-1}\partial \log f_{ij}/\partial z$$

are meromorphic in M. This proves Theorem J in [1]. It is classical that every meromorphic function in $D(\tau)$ which is automorphic under $G(\tau)$ can be expressed rationally in terms of the functions f_{ij}, f_{ijk} (and even in terms of the f_{ij} alone if $S(\tau)$ is not hyperelliptic). Thus we obtain a proof of Theorem 4 in [3] which asserts the existence of finitely many meromorphic functions of the moduli and of an additional complex variable, which uniformize simultaneously all algebraic curves of genus $g>1$.

(b) Let us choose $(2q-1)(g-1)$ elements of $\boldsymbol{W}_q$, $q>1$ (or g elements of $\boldsymbol{W}_1$) which are linearly independent for $\tau=\tau_0$, and let $w(z, \tau)$ denote their Wronskian with respect to z. For a fixed τ close to τ_0 the zeros of $w(z, \tau)$ are precisely the Weierstrass points of $S(\tau)$, in the classical sense if $q=1$, in the sense of Petersson if $q>1$ (cf. the definition in [4]). Since w is a holomorphic function in M we conclude that *the Weierstrass points on a closed Riemann surface depend holomorphically on the moduli* (cf. Rauch [6], Röhrl [3]).

(c) Now let $w(z, \tau)$ denote the Wronskian of an arbitrary set of dim $W_q(\tau)$ elements of $\boldsymbol{W}_q$ and let N denote the set of those $\tau \in T$ for which $w(z, \tau)\equiv 0$. If z_0 is not a Weierstrass point of $S(\tau_0)$, then there is a neighborhood of τ_0 in which the points of N are precisely the zeros of $w(z_0, \tau)$. We conclude that *N is either empty, or the whole domain T, or an analytic subset of T of codimension* 1.

(d) Let H denote the set of those $\tau \in T$ for which $S(\tau)$ is hyperelliptic. For $\tau \in T-H$ every element of $W_q(\tau)$ can be written as a homogeneous polynomial in the p_j (M. Noether). For $\tau \in H$ the subspace of $W_q(\tau)$ consisting of homogeneous polynomials in elements of $W_1(\tau)$ has dimension $q(g-1)+1$. But H is an analytic subvariety of T of dimension $2g-1$, so that, noting (c), we obtain the following complement to Noether's theorem: *for $g>3$ and $q>1$ there exist no fixed set of $(2q-1)(g-1)$ homogeneous polynomials of degree q in normalized Abelian differentials of the first kind which spans the space of* *holomorphic differentials of dimension q on all nonhyperelliptic closed Riemann surfaces of genus g.*

(e) *The mapping* $\tau \to Z(\tau)$ of the Teichmüller space into the Siegel space *is holomorphic.* This follows at once from Theorem III, and also by using the coordinates in T defined in [1] in conjunction with Rauch's variational formulas [5]. These formulas also show that the mapping of T into a $(3g-3)$-dimensional subspace of the Siegel space

$$\tau \to \left\{ \sum_{i,k=1}^{g} \gamma_{j,ik} Z_{ik}(\tau),\ j = 1, \cdots, 3g-3 \right\}$$

is one-to-one near a point τ_0 if and only if the $3g-3$ functions

$$\sum_{i,k=1}^{g} \gamma_{j,ik} p_i(z, \tau_0) p_k(z\ \tau_0)$$

are linearly independent. This shows that near every nonhyperelliptic surface a properly chosen set of $3g-3$ periods Z_{ik} can serve as a set of local moduli (Rauch). On the other hand, (d) implies a complement to Rauch's theorem: *no fixed set of $3g-3$ linear combinations of periods can serve as a set of moduli near every nonhyperelliptic closed Riemann surface of genus $g > 3$.*

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