ON SOLUTIONS OF RIEMANN'S FUNCTIONAL EQUATION

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1. Let $\{\lambda_n\}$, $\{\mu_n\}$ $(n \geqq 1)$ be two given sequences of positive numbers increasing to infinity, and let $\delta > 0$. We call the triplet $\{\delta, \lambda_n, \mu_n\}$ a *label*. If s is a complex variable, $s = \sigma + i\tau$, we speak of a solution of Riemann's functional equation

$$(1.1) \qquad \pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\phi(s) = \pi^{-(\delta-s)/2}\Gamma\left\{\frac{1}{2}(\delta - s)\right\}\psi(\delta - s),$$

pertaining to the label $\{\delta, \lambda_n, \mu_n\}$, if there exist two Dirichlet series $\phi(s) = \sum a_n\lambda_n^{-s}$, $\psi(s) = \sum b_n\mu_n^{-s}$ (a_n and b_n complex) which do not vanish identically, and which admit finite abscissae of absolute convergence, and a function $\chi(s)$ which is holomorphic and uniform in a domain $|s| > R$, such that $\lim_{|\tau|\to\infty}\chi(\sigma + i\tau) = 0$ uniformly in every segment $\sigma_1 \leqq \sigma \leqq \sigma_2$, and such that, for some pair of real numbers α, β, we have

$$\chi(s) = \begin{cases} \pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\phi(s), & \text{for } \sigma > \alpha, \\ \pi^{-(\delta-s)/2}\Gamma\left\{\frac{1}{2}(\delta - s)\right\}\psi(\delta - s), & \text{for } \sigma < \beta. \end{cases}$$

In three papers published recently, Bochner and Chandrasekharan [2], Chandrasekharan and Mandelbrojt [3], and Kahane and Mandelbrojt [4], have studied the problem of finding an upper bound for the number of linearly independent solutions of equation (1.1). Their results enable one to establish in certain cases a unique solution, and in certain others to deduce that the sequences $\{\lambda_n\}$, $\{\mu_n\}$ are periodic. In this note, which is a sequel to [3], we shall consider certain simple conditions which would ensure that $\delta = 1$. Let

$$D^{\mu} = \limsup\,(n/\mu_n), \qquad h_{\mu} = \liminf\,(\mu_{n+1} - \mu_n).$$

We prove the following results.

THEOREM 1. *If $h_\lambda \cdot h_\mu = 1$, δ is an odd integer, and equation* (1.1) *has a solution, then $\lambda_{n+1} - \lambda_n = h_\lambda$, and $\mu_{n+1} - \mu_n = h_\mu$, for every $n \geqq 1$. In particular, if $h_\lambda = h_\mu = 1$, δ is an odd integer, and equation* (1.1) *has a solution, then $\lambda_{n+1} - \lambda_n = 1$, and $\mu_{n+1} - \mu_n = 1$ for every $n \geqq 1$.*

THEOREM 2. *If* $h_\mu > 0$, δ *is an odd integer*, $b_n = O(1)$, *and equation* (1.1) *has a solution, then* $\delta = 1$.

THEOREM 3. *Let* $h_\mu > 0$, *and let* δ *be an odd integer. If simultaneously*, $(\sum a_n \lambda_n^{-s}, \sum b_n \mu_n^{-s})$ *is a solution of equation* (1.1) *with the label* $(\delta, \lambda_n, \mu_n)$, *and* $(\sum c_n \lambda_n'^{-s}, \sum d_n \mu_n^{-s})$ *is a solution with the label* $(\delta, \lambda_n', \mu_n)$ *for some* (λ_n'), *and* $(\sum e_n \lambda_n''^{-s}, \sum b_n d_n \mu_n^{-s})$ *is also a solution with the label* $(\delta, \lambda_n'', \mu_n)$ *for some* (λ_n''), *where* $(b_n/d_n) = o(\mu_n)$; *then* $\delta = 1$.

2. For the proof of these theorems we require a number of lemmas.

LEMMA 1. *Equation* (1.1) *implies, for a sufficiently large integer r, the following relation*:

$$\Gamma\left\{\frac{1}{2}(\delta+1)\right\}\pi^{-(\delta+1)/2}\sum_{n=1}^{\infty} a_n\left[\frac{d^{2r}}{ds^{2r}}\frac{s}{(s^2+\lambda_n^2)^{(\delta+1)/2}}\right] - K_r(s) = (2\pi)^{2r}\sum_{n=1}^{\infty} b_n\mu_n^{2r}\exp(-2\pi\mu_n s), \tag{2.1}$$

for Re $s > 0$, *where* $K_r(s)$ *is holomorphic on the surface on which* log s *is defined, and* $K_r(s) = O(|s|^{-\epsilon})$, $\epsilon > 0$, *as* $s \to \infty$ *in any angle* $|\arg s| \leqq \theta_0$.

This has been proved by Bochner and Chandrasekharan [Theorem 2.1, p. 344]. By the definition of functional equation (1.1) it follows that the Dirichlet series on the right of (2.1) converges absolutely for $\sigma > 0$, and from (2.1) it follows that the singularities of its sum-function are situated symmetrically on the imaginary axis $\sigma = 0$, at the points $(\pm i\lambda_n)$, and also possibly at the origin, which we may, for convenience, designate as λ_0.

LEMMA 2. *If* $D^\mu < \infty$, *and equation* (1.1) *has a solution, then* $D^\lambda \cdot D^\mu \geqq 1$, *and* $h_\lambda \cdot h_\mu \leqq 1$. (*With the understanding that if* $D^\mu = 0$, *then* $D^\lambda = +\infty$.)

This is an immediate consequence of a theorem of Chandrasekharan and Mandelbrojt [**3**, Theorem 1, p. 289] which implies [loc. cit., p. 290, ll. 6–9] that if $D^\mu < \infty$, and equation (1.1) is satisfied, then $\lambda_{n+1} - \lambda_n \leqq D^\mu$ for every $n \geqq 1$, that is, $\lambda_n \leqq n \cdot D^\mu$, or $n/\lambda_n \geqq 1/D^\mu$, or $D^\lambda \cdot D^\mu \geqq 1$. Since we have $D^\mu \cdot h_\mu \leqq 1$, it follows that $h_\lambda \cdot h_\mu \leqq 1$.

LEMMA 3. *If* $h_\mu > 0$, δ *is an odd integer, and equation* (1.1) *has a solution, then* $\delta = 1$ *or* 3.

This is a result of Kahane and Mandelbrojt [**4**, Theorem 3, pp. 71–72].

LEMMA 4. *If $h_\mu > 0$, and $\delta = 1$ or 3, and equation* (1.1) *has a solution, then $\mu_{n+1} - \mu_n \geqq h_\mu$. And for $\sigma < 0$, the analytic continuation of the series*

$$\Psi(s) = \begin{cases} \sum b_n \exp(-2\pi\mu_n s) & \text{if } \delta = 1, \\ \sum b_n \mu_n^{-1} \exp(-2\pi\mu_n s), & \text{if } \delta = 3, \end{cases}$$

which is a uniform function, is given by the series $-\sum_1^\infty b_n \exp(+2\pi\mu_n s)$, and the only singularities of $\Psi(s)$ are simple poles at the points $\pm i\lambda_n$, $n = 0, 1, 2, \cdots$.

A result proved earlier by Chandrasekharan and Mandelbrojt [**3**, Theorem 3, p. 292] gives the Dirichlet series representation of $\Psi(s)$ in the negative half-plane as $\sum c_n \exp(2\pi\mu_n' s)$ but it is easy to see that $c_n = -b_n$, and $\mu_n = \mu_n'$, if one observes that by Agmon's theorem, used in that proof, the origin is a simple pole for the residual function $K_r(s)$ in (2.1). This fact is also obvious from the paper by Kahane and Mandelbrojt [**4**].

LEMMA 5. *If $h_\mu > 0$, and $f(s) = \sum_0^\infty B_n \exp(-2\pi\mu_n s)$ has $\sigma = 0$ as its abscissa of absolute convergence, and the only singularities of $f(s)$ on a segment of the imaginary axis of length greater than h_μ^{-1} are poles of greatest order q, then $B_n = O(\mu_n^{q-1})$.*

This is a tauberian theorem of S. Agmon [**1**, Theorem 4.3(C)].

LEMMA 6. *If $D^\mu < \infty$, and $b_n = O(\mu_n^{q-1})$, then for $\sigma > 0$, we have*

$$f(s) \equiv \sum_1^\infty b_n \exp(-2\pi\mu_n s) = O(\sigma^{-q}).$$

If in the hypothesis we have $b_n = o(\mu_n^{q-1})$, then the conclusion is $f(s) = o(\sigma^{-q})$.

(i) Since $D^\mu < \infty$, we have $\mu_n > Ln$ for every n, where L is some constant. Now, for $\sigma > 0$, we have

$$|f(s)| \leqq C \cdot \sum_1^\infty \mu_n^{q-1} \exp(-2\pi\mu_n\sigma)$$

$$\leqq C \cdot (2\pi\sigma)^{1-q} \sum_1^\infty (2\pi\mu_n\sigma)^{q-1} \exp(-2\pi\mu_n\sigma).$$

The term $(2\pi\mu_n\sigma)^{q-1} \exp(-2\pi\mu_n\sigma)$ decreases (as μ_n increases), when $2\pi\mu_n\sigma > q-1$. Let n_σ be the smallest n for which we have $2\pi Ln\sigma > q-1$; in other words, for $n = 1, \cdots, n_\sigma - 1$, we have $2\pi Ln\sigma \leqq q-1$. Then

$$\sum_{n_\sigma}^{\infty} (2\pi\mu_n\sigma)^{q-1} \exp(-2\pi\mu_n\sigma) \leqq \sum_{n_\sigma}^{\infty} (L\cdot 2\pi n\sigma)^{q-1} \exp(-2\pi L n\sigma)$$
$$= O\left(\sigma^{q-1}\sum_{n_\sigma}^{\infty} n^{q-1}\exp(-2\pi L n\sigma)\right)$$
$$= O(\sigma^{-1})$$

while

$$\sum_{1}^{n_\sigma-1} (2\pi\mu_n\sigma)^{q-1} \exp(-2\pi\mu_n\sigma) \leqq \max_{x\geqq 0}\left[x^{q-1}e^{-x}\right]\cdot(n_\sigma - 1)$$
$$\leqq K\cdot(n_\sigma - 1) = O(\sigma^{-1}).$$

Hence $f(s) = O(\sigma^{-q})$.

(ii) In case $b_n = o(\mu_n^{q-1})$, let n_σ be the smallest n such that $(2\pi n\sigma) > (q-1)\sigma^{1/2}$. Then, as before,

$$\left|\sum_{1}^{n_\sigma-1} b_n \exp(-2\pi\mu_n s)\right| = O(\sigma^{1-q}(n_\sigma - 1)) = O(\sigma^{1/2-q}),$$

and, since $n_\sigma \to \infty$, as $\sigma\to 0$, we have

$$\left|\sum_{n_\sigma}^{\infty} b_n \exp(-2\pi\mu_n s)\right| = o(\sigma^{1-q})\cdot\left|\sum_{n_\sigma}^{\infty}(2\pi\mu_n\sigma)^{q-1}\exp(-2\pi\mu_n\sigma)\right|$$
$$= o(\sigma^{-q}).$$

Hence $f(s) = o(\sigma^{-q})$.

3. We shall now indicate the proofs of Theorem 1 to 3.

Proof of Theorem 1. We remark that by Lemma 2, we have $h_\lambda\cdot h_\mu \leqq 1$. If $h_\lambda\cdot h_\mu = 1$, then we have $h_\lambda > 0$, and $h_\mu > 0$, so that $D^\lambda < \infty$, and $D^\mu < \infty$. Hence, as in the proof of Lemma 2, we have $\mu_{n+1} - \mu_n \leqq D^\lambda \leqq h_\lambda^{-1} = h_\mu$, and $\lambda_{n+1} - \lambda_n \leqq D^\mu \leqq h_\mu^{-1} = h_\lambda$. Since δ is odd, we have, by Lemma 3, $\delta = 1$ or 3. Now, by the first part of Lemma 4, we have $\mu_{n+1} - \mu_n \geqq h_\mu$, and $\lambda_{n+1} - \lambda_n \geqq h_\lambda$, which lead to the desired result.

Proof of Theorem 2. By Lemma 3, we have $\delta = 1$ or 3. We shall show that the case $\delta = 3$ is incompatible with the hypotheses. Consider the series $f(s) = \sum b_n\mu_n^{2r}\exp(-2\pi\mu_n s)$ in Lemma 1. Since $b_n = O(1)$, we have, by Lemma 6, $f(s) = O(\sigma^{-2r-1})$, for $\sigma > 0$. On the other hand, in a neighborhood of a pole, say $s = i\lambda_n$, $n \geqq 1$, we have $|f(s)| > c\cdot|\sigma|^{-p}$, where p is the order of the pole, hence an integer, with $p = (1/2)(\delta+1) + 2r$. For these two estimates to be compatible, we should have $\delta = 1$.

Proof of Theorem 3. It is sufficient to show that $\delta = 3$ is impossible. If $\delta=3$, then by Lemmas 5 and 1, we have $b_n=O(\mu_n)$, $d_n=O(\mu_n)$ and $b_nd_n=O(\mu_n)$.But by hypothesis, $|b_n|\leqq\epsilon_n\cdot|d_n|\cdot\mu_n$, where $\epsilon_n>0$, and $\epsilon_n\to 0$ as $n\to\infty$. That is, $|b_nd_n|\geqq|b_n|^2\cdot(\mu_n\epsilon_n)^{-1}$. We now observe that $b_n=o(\mu_n)$ is impossible, because otherwise, by Lemma 6(ii), we should have $f(s)=o(\sigma^{-2})$, which contradicts the fact that $|f(i\lambda_n+\sigma)|>c\cdot\sigma^{-2}$ for $\sigma>0$. Hence there exists a sequence (n_j) such that $|b_{n_j}|>\epsilon_{n_j}^{1/3}\cdot\mu_{n_j}$, which, together with the inequality for b_nd_n obtained above, yields $|b_{n_j}\cdot d_{n_j}|\geqq\epsilon_{n_j}^{2/3}\cdot\mu_{n_j}^2\cdot(\mu_{n_j}\cdot\epsilon_{n_j})^{-1}\geqq\mu_{n_j}\cdot\epsilon_{n_j}^{-1/3}$. But this contradicts the fact that $b_nd_n=O(\mu_n)$.

References

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