NOTE ON THE COBORDISM RING

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Cobordism is an equivalence relation among compact differentiable manifolds which can be roughly described by: two manifolds are cobordant if together they form the boundary of a differentiable manifold with boundary. Disjoint union and topological product of manifolds induce addition and multiplication operators with respect to which the equivalence classes form a ring graded by dimension. In fact we obtain two rings, depending whether we consider oriented or nonoriented manifolds, and denote them by Ω and $\mathfrak{N}$ respectively. (For precise definitions see [5].)

Thom has shown that $\mathfrak{N}$ is a ring of polynomials mod 2, with one generator x_i in each dimension not of the form 2^i-1, that two nonoriented manifolds are cobordant if and only if they have the same Stiefel numbers, and that x_{2n} can be chosen as the class of the real projective space $P_{2n}(R)$. In [1] Dold gave orientable manifolds V_{2n-1} which can be taken as the odd-dimensional generators.

Much work has been done on the determination of Ω: it has been known for some time that Pontrjagin numbers are invariants of cobordism class, and Thom has shown that $\Omega\otimes Q$ is a polynomial ring, that complex projective spaces $P_{2n}(C)$ can be taken as its generators, that two oriented manifolds determine the same class in it if and only if they have the same Pontrjagin numbers, and that the cobordism groups (graded components of Ω) are all finitely generated Abelian groups. Rohlin studied in [3] the natural map $r\colon \Omega\to\mathfrak{N}$ obtained by ignoring orientation, and Milnor in [2] showed that Ω has no odd torsion, and that the torsion-free part is a polynomial algebra. These results may be completed by the following, which now determine the algebraic structure of Ω (and incidentally contradict a statement in [4] on which the main theorems of that paper seem to depend):

(1) The cobordism ring Ω contains no elements of order 4.

(2) Two oriented manifolds are cobordant if and only if all their Pontrjagin and Stiefel numbers are the same.

(3) There is a polynomial subalgebra $\mathfrak{W}$ of $\mathfrak{N}$ containing $r(\Omega)$, and a map $\partial\colon \mathfrak{W}\to\Omega$ such that the following triangle is exact:

$$\begin{array}{ccc} \Omega & \xrightarrow{\;2\;} & \Omega \\ & {\scriptstyle\partial}\nwarrow \qquad \swarrow{\scriptstyle r} & \\ & \mathfrak{W} & \end{array}$$

We define $\mathfrak{W}$ as the subset of $\mathfrak{N}$ of classes containing a manifold M such that the first Stiefel-Whitney class W_1 is the restriction of an integer class, and thus corresponds to a map $f: M \to S^1$. Regularise f; then $V = f^{-1}(0)$ is a submanifold of dimension $n-1$ of M, with trivial normal bundle, and homology class mod 2 dual to W_1. The orientation bundle of M is induced from the double covering of S^1, so a neighbourhood of V in M is orientable, hence so is V. Let W_ω denote the product $W_{a_1} W_{a_2} \cdots W_{a_r}$ of Stiefel-Whitney classes. Then if $i: V \subset M$, we have

$$[V, W_\omega(V)] = [V, i^* W_\omega(M)] = [i_* V, W_\omega(M)]$$
$$= [(M \cap W_1), W_\omega(M)] = [M, W_1 W_\omega(M)]$$

since the class of V is dual to W_1. Hence the Stiefel numbers of M determine those of V, so the cobordism class of M determines that (in $\mathfrak{N}$) of V. I denote this operation by ∂'.

A manifold V can be obtained by the above construction if and only if $2V$ bounds orientably. For if V can be so obtained, we cut M along V and find a manifold M' with boundary $2V$ (since the normal bundle of V in M is trivial) and M' is orientable; and this argument is reversible.

The set of manifolds M is closed under disjoint union and product, so gives a subalgebra $\mathfrak{W}$ of $\mathfrak{N}$. This contains (i) Dold manifolds V_{2i-1} $(i \neq 2^j)$, which are orientable and of order 2 in Ω by [1], (ii) manifolds M_{2i} corresponding to them by the above construction: these are explicitly known, so we can verify directly that $s_{2i}(M_{2i}) = 1$, so that the M_{2i} can be taken as generators in $\mathfrak{N}$, and (iii) spaces $P_{2^n}(C)$. Since, by a computation with Stiefel numbers, $P_n(C)$ and $(P_n(R))^2$ are cobordant, all these generate a polynomial algebra. We prove this algebra equal to $\mathfrak{W}$ by showing that it has the same number of elements in each dimension as there are classes in $\mathfrak{N}$ none of whose nonzero Stiefel numbers has a factor W_1^2 (note that if W_1 comes from an integral class, $O = Sq^1 W_1 = W_1^2$).

For this we apply Thom theory. Now $H^{n+k}(M(O_k), Z_2)$ can be regarded if $n < k$ as the space of polynomials of degree n in the Stiefel classes, so that a manifold of dimension n induces on it a linear functional, which annihilates decomposable elements of $H^*(M(O_k), Z_2)$, considered as a module over the Steenrod algebra $\mathfrak{A}_2$. Thom showed that the algebra of the W_i is a free $\mathfrak{A}_2$-module; the determination of $\mathfrak{W}$ now follows from the

LEMMA. *The ideal generated by W_1^2 in this algebra is a free $\mathfrak{A}_2$-module, and a direct summand.*

We have now computed $\mathfrak{W}$ and the values of ∂' on the generators. Its other values follow, since it is a derivation, as we show by computing with Stiefel numbers. We deduce that Ker $\partial'/\text{Im}\,\partial'$ is a polynomial algebra, generated by the squares of even-dimensional generators of $\mathfrak{N}$. The Pontrjagin numbers, reduced mod 2, give linear functionals on $\mathfrak{N}$, and thus form the dual vector space to Ker $\partial'/\text{Im}\,\partial'$ (this holds since $r(p_i) = W_{2i}^2$).

We now need the following result of Rohlin [3] (there exists also a proof due to Dold).

LEMMA. *The sequence* $\Omega \xrightarrow{2} \Omega \xrightarrow{r} \mathfrak{N}$ *is exact.*

Let c be a torsion element of Ω_m, of maximal order, which by Milnor's result is of the form 2^x, and suppose if possible $1<x$. rc is orientable, so in Ker ∂'. But c is a torsion element, so all Pontrjagin numbers vanish on it, so rc is in Im ∂'. Hence there is a class c', of order 2, such that $r(c-c')=0$. Since $1<x$, $c-c'$ has the same order as c. Now by the lemma, $c-c'=2d$, so d has order 2^{x+1}, contrary to the maximality of c. This proves (1). (2) follows by the lemma, since the only elements in Ker $r=$ Im 2 occur in free groups, so have non-zero Pontrjagin numbers. But now, since the Pontrjagin numbers of V are zero (V having order 2), and its Stiefel numbers are determined by those of M, its class in Ω is determined by the class of M, and hence ∂ is defined. Moreover, classes in Im ∂' are all orientable, and cosets in Ker $\partial'/$Im ∂' are in fact all represented by polynomials in complex projective spaces, so classes in Ker $\partial=$ Ker ∂' are all orientable. Assembling our results, (3) now follows.

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