FIXED POINTS OF ELEMENTARY COMMUTATIVE GROUPS

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Communicated by Leo Zippin, June 24, 1959

In this Note G is always a compact Lie group. A G-space is a topological space on which G acts continuously. We shall be mainly concerned with the case where G is an *elementary commutative p-group* (p prime or zero), that is, a direct product of a finite number of copies of the circle group T^1 if $p=0$ or of the cyclic group Z_p of order p if p is prime. In this study, a basic role is played by the space X_G, defined in §1. The proofs and additional results are given in the Notes of a seminar on transformation groups, to be published in the Annals of Mathematics Studies.

Although it is not always essential, we assume for simplicity that X is locally compact. $H_c^i(X; L)$ (resp. $H^i(X; L)$) is the ith cohomology group of X with compact (resp. closed) supports, coefficients in L. $\dim_L X$ (resp. $\dim_p X$) is the cohomological dimension [**6**] with respect to L (resp. a field K_p of characteristic p). The orbit space of X is denoted by X'; $\pi_X\colon X \to X'$ is the canonical projection, $G_x = \{g \in G,\ g\cdot x = x\}$ the stability group of X and $F(H; X)$ the fixed point set of a subgroup H of G.

1. The space X_G. Let X, Y be two G-spaces. The twisted product $X \times_G Y$ is the orbit space of $X \times Y$ under the "diagonal" action $g(x, y) = (g\cdot x, g\cdot y)$. The projections on the two factors induce maps π_1, π_2 of $X \times_G Y$ onto X' and Y' respectively. It is easily seen that given $x' \in X'$ and $x \in \pi_X^{-1}(x')$, the subspace $\pi_1^{-1}(x')$ may be identified with Y/G_x and similarly for π_2. Also, π_2 is a bundle map when Y is a principal G-bundle. We specialize Y to be a universal bundle E_G for G, hence $Y' = E_G/G$ is a classifying space B_G for G.

1.1. LEMMA. *Let $X_G = X \times_G E_G$. Then X_G has projections π_1, π_2 on X' and B_G respectively. π_2 is the projection in a fibre bundle with structural group G and typical fibre X. Let $x' \in X'$, then $\pi_1^{-1}(x')$ may be identified with B_{G_x} ($x \in \pi_X^{-1}(x')$). If G acts trivially on X, then $X_G = X \times B_G$. If $x \in F$, then $\pi_1^{-1}(x) = x \times B_G$ is a cross section for π_2.*

This space has occurred earlier in special cases (see [**2**; **7**; **8**] for instance). For discrete G, an algebraic analogue may be found in [**11**, Chapter V].

1.2. LEMMA. *Assume that* $\dim_L X$ *is finite and that* $H^*(B_{G_x}; L)$ *is trivial for* $x \notin F$. *Then the restriction map* $H_c^*(X_G; L) \to H_c^*(F \times B_G; L)$ *induced by the inclusion* $F_G \to X_G$ *is an isomorphism in degrees* $> \dim_L X$.

This applies in particular when $G = T^1$, $L = K_0$, or $G = L = Z_p$ (p prime) and yields easily the Smith theorem on homology spheres [13], its analogue for circle groups [8], and the dimensional parity theorems of Floyd [9] and Liao [12]. Also, dimensional parity holds when $G = Z_4$, $L = Z_2$ and G acts freely outside F.

2. **Fixed point theorems.** Elementary commutative p-group will be abbreviated by $[p]$-group. In this section, G is a $[p]$-group, X is a G-space which is compact, connected, of finite dimension over K_p, and the number of distinct stability groups of G on X is finite.

2.1. THEOREM. *Assume that* $\dim H^*(X; K_p)$ *is finite, and that* X *is totally nonhomologous to zero* mod p *in* X_G. *Then* $\dim H^*(F; K_p) = \dim H^*(X; K_p)$. *In particular* $F \neq \phi$.

It is easy to see that if G is any p-group, we have $\dim H^*(X; K_p) \geqq \dim H^*(F; K_p)$ provided G acts trivially on $H^*(X; K_p)$, so that the main point here is the reverse inequality. The latter is proved by a discussion of the Fary spectral sequence of π_1, for a suitable family of closed sets. A similar argument proves the:

2.2. THEOREM. *Assume that* X *is a cohomology n-sphere* mod p. *Let* G_i ($i \geqq 1$) *be the different subgroups of* G *which have index* p *if* $p \neq 0$, *or which are connected and of codimension* 1 *if* $p = 0$. *Let* n_i (*resp.* r) *be the integer such that the fixed point set of* G_i (*resp.* G) *is a cohomology* n_i-(*resp.* $r-$) *sphere* mod p *by the Smith theorem. Then* $n - r = \sum_i (n_i - r)$.

3. **Applications.** A compact connected manifold of dimension $2n$ is homologically Kählerian if there exists a class $Q \in H^2(X; K_0)$ such that the multiplication by Q^{n-s} is an isomorphism of $H^s(X; K_0)$ onto $H^{2n-s}(X; K_0)$ $(0 \leqq s \leqq n)$.

3.1. THEOREM. *Let* G *be a toral group acting on a compact connected homologically Kählerian manifold.* (a) *if* $F \neq \phi$, *then* $\dim H^*(F; K_0) = \dim H^*(X; K_0)$, (b) *if* $H'(X; K_0) = 0$, *then* $F \neq \phi$. (c) *If* X *has no torsion for some prime* p, *then* F *has no* p-*torsion.*

This follows from the above and Theorems II. 1.1, II. 1.2 of [1]. These results form a topological counterpart to the (more precise) ones obtained by Frankel [10] for toral groups of isometries on Kählerian manifolds.

3.2. THEOREM. *Let K be a compact connected Lie group, U a closed subgroup, and p be a prime. Assume that $H^*(K/U; K_p)$ is equal to its characteristic algebra* [2, §18]. *Then every* $[p]$*-subgroup G of K is conjugate to a subgroup of U. In particular, if K or B_K has no p-torsion, then every* $[p]$*-subgroup of K lies in a torus of K.*

This is proved by showing that one can apply 2.1 to G operating by left translations on K/U or respectively on K/T, where T is a maximal torus of K. We mention that there is a converse to the second assertion of 3.2. For further results concerning p-torsion and $[p]$-subgroups of compact Lie groups, see a forthcoming Note of the author.

3.3. THEOREM. *Let G be a* $[p]$*-group (p prime) acting on X. Assume that $H^*(X; K_p) = K_p[x]/(x^{s+1})$, (the degree k of x being even if $p \neq 2$). Then* $\dim H_c^*(F; K_p) = s+1$ *in each of the following cases:* (a) $(p, s+1) = 1$, (b) $k=4$, $p \neq s+1$, $p \neq 2$, (c) *X is the quaternionic projective space of real dimension* $4s$ $(s \geqq 1)$ *and* $p \neq 2$.

This follows from 2.1 and from the fact, that, in the cases (a), (b), (c), the space X is totally nonhomologous to zero mod p in any fibre bundle whose structural group acts trivially on $H^*(X; K_p)$. Examples of fixed point free $[p]$-groups on complex or quaternionic projective spaces or on the Cayley plane show that 3.3 probably cannot be improved significantly.

4. **Cohomology manifolds.** Given open subsets $U \subset V$ of X, we denote by j_{VU}^i the natural homomorphism $H_c^i(U; L) \to H_c^i(V; L)$ and by j_{VU}^* the sum of the j_{VU}^i. A space X is a *cohomology n-manifold* (an $n-cm$) over the principal ideal domain L of coefficients if it has finite dimension over L and if for each $x \in X$ there is an open neighborhood U_x of x and a free submodule of rank one A_x of $H_c^n(U_x; L)$ such that every $y \in U_x$ has a basic system of open neighborhoods V for which $A_x = \operatorname{Im} j_{U_x V}^*$. It is orientable if U_x may be taken as the connected component of x.

This definition corresponds to the locally orientable generalized manifold of Wilder [**3**; **4**]. It is equivalent to that of Yang [**14**]. A $n-cm$ is cohomologically locally connected over L [**3**], and of dimension n over L. If it is orientable and connected, then $H_c^n(X; L) = L$. The Poincaré duality, proved in [**3**] when L is a field, is extended to the general case in [**4**]. In particular, if X is connected and orientable, there is an exact sequence

$$0 \to \operatorname{Ext}(H_c^{i+1}(X; L), L) \to H^{n-i}(X; L) \to \operatorname{Hom}(H_c^i(X; L), L) \to 0$$

where $H^i(X; L)$ denotes cohomology with closed supports.

4.1. PROPOSITION. *Let X be a G-space which is a connected $n-cm$ over L. Assume that $H^*(B_{G_x}; L)$ is trivial for $x \notin F$. Let $U = U_0 \supset \cdots \supset U_n = V \supset \cdots \supset U_{2n}$ be a sequence of connected orientable invariant neighborhoods such that $j^i_{U_j U_{j+1}}$ is zero for $i \neq n$ and an isomorphism for $i = n$ $(0 \leqq j \leqq 2n-1)$. Then* $\operatorname{Im} j^i_{U \cap F, V \cap F} \cong H_c^{n-i}(B_G; H_c^n(U; L))$ *and is a direct summand.*

Here is meant the cohomology of B_G with respect to the sheaf defined by the nth cohomology groups of the fibres in the fibering of U_G over B_G. This result is the main tool in proving the local theorems: F is a cohomology manifold over L in the cases $G = L = Z_p$ (Smith) or $G = T^1$, L a field or the integers (Conner-Floyd), and is orientable if X is. There are also dimensional parity theorems paralleling those mentioned at the end of §1, the first two of which were obtained first by Bredon [5] to which one may add the following: If $G = T^1$, $L = Z_2$, then the dimension over Z_2 of any component Y of the fixed point set X of the element of order 2 in G has the same parity as $\dim_2 X$ provided Y contains a point fixed under G. Finally, 2.2 has also a local analogue:

4.2. THEOREM. *Let G be a $[p]$-group, X be a $n-cm$ over K_p and $x \in F$. The subgroups G_i being defined as in 2.2, let n_i (resp. r) be the dimension over K_p of the component of $F(G_i; X)$, (resp. $F(G; X)$) passing through x. Then $n - r = \sum_i (n_i - r)$.*

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