ON A CONJECTURE CONCERNING SCHLICHT FUNCTIONS[1]

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Many years ago and independently of each other S. Mandelbrojt and M. Schiffer were led to the following conjecture, which has appeared in print only recently [2, p. 326]:

CONJECTURE M. S. *If two power series* $\sum_1^\infty a_\nu z^\nu$, $\sum_1^\infty b_\nu z^\nu$ *are schlicht in the unit circle, then also the power series*

$$\sum_1^\infty \frac{a_\nu b_\nu}{\nu} z^\nu$$

is schlicht in the unit circle.

This will be disproved in the following lines. Let D be the image of the unit circle by $w = \sum_1^\infty a_\nu z^\nu$. We denote by the symbols S, Σ and K the classes of such power series for which D is schlicht, schlicht and star-shaped, schlicht and convex, respectively. Evidently $K \subset \Sigma \subset S$.

Observe now that $\sum_1^\infty z^\nu \in K$. By a recent result concerning de la Vallée Poussin means [2, p. 298] we conclude that

$$\sum_1^n \binom{2n}{n+\nu} z^\nu \in K, \qquad (n = 1, 2, \cdots),$$

and therefore [2, Lemma 5, p. 321] that

$$\sum_1^n \nu \binom{2n}{n+\nu} z^\nu \in \Sigma \subset S.$$

Applying the Conjecture M. S. to this special polynomial and an arbitrary power series we obtain the following

COROLLARY OF THE CONJECTURE M.S. *If* $f(z) = \sum_1^\infty a_\nu z^\nu \in S$ *then also*

$$\sum_1^n \binom{2n}{n+\nu} a_\nu z^\nu \in S.$$

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In other words: The de la Vallée Poussin means of schlicht functions are also schlicht. However, this corollary is now easily disproved as follows:

We appeal to a result of C. Loewner [1, pp. 117, 118, and 120]: To every given function $\kappa(\tau)$ which is continuous for $\tau \geqq 0$ and such that $|\kappa(\tau)| = 1$, there corresponds a power series $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ which is in S and is such that

$$a_2 = -2\int_0^\infty \kappa(\tau)e^{-\tau}d\tau,$$

$$a_3 = 4\left(\int_0^\infty \kappa(\tau)e^{-\tau}d\tau\right)^2 - 2\int_0^\infty (\kappa(\tau))^2 e^{-2\tau}d\tau.$$

We now select

$$\kappa(\tau) = e^{-i\gamma\tau}, \qquad (\gamma \text{ real constant} \neq 0).$$

The integrals are easily evaluated and we find

$$f(z) = z - \frac{2}{1 + i\gamma} z^2 + \frac{3 - i\gamma}{(1 + i\gamma)^2} z^3 + \cdots \in S.$$

Applying the Corollary of the Conjecture M.S. for $n = 3$ we conclude that the cubic polynomial

$$P(z) = 15z + 6a_2 z^2 + a_3 z^3 \in S.$$

But then *the quadratic polynomial*

$$\frac{1}{3}(1 + i\gamma)^2 P'(z) = (3 - i\gamma)z^2 - 8(1 + i\gamma)z + 5(1 + i\gamma)^2$$

can not have any zeros in the interior of the unit circle.

On the other hand we find that

$$\zeta = \frac{1 + i\gamma}{3 - i\gamma}(4 - (1 + 5i\gamma)^{1/2})$$

is one of the two zeros of this quadratic; ζ is regular for all real γ and we find its Taylor expansion at the origin to be

$$\zeta = 1 + \frac{i}{2}\gamma - \frac{9}{24}\gamma^2 + \cdots, \qquad (|\gamma| < 1/5).$$

Now

$$|\zeta|^2 = \left(1 - \frac{9}{24}\gamma^2 + \cdots\right)^2 + \left(\frac{\gamma}{2} + \cdots\right)^2 = 1 - \frac{1}{2}\gamma^2 + \cdots,$$

showing that $|\zeta| < 1$ provided that γ is sufficiently small. This contradicts our last italicized statement and completes our proof.

For the discussion of a conjecture of Polya and Schoenberg obtained from the Conjecture M.S. by replacing in its statement the term "schlicht" by "star-shaped," we refer to [2, pp. 324–334].

References

1. C. Loewner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. vol. 89 (1923) pp. 103–121.

2. G. Polya and I. J. Schoenberg, *Remarks on de la Vallée Poussin means and convex conformal maps of the circle*, Pacific J. Math. vol. 8 (1958) pp. 295–334.

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