RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

AN n-DIMENSIONAL ANALOGUE OF THE CREMONA-CLEBSCH THEOREM

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Communicated by S. Bochner, December 26, 1958

1. Given a Cremona transformation T between two complex projective planes P and P', the exceptional elements in both planes form closed subsets $M \subset P$, $M' \subset P'$ such that T induces an analytical homeomorphism of the open sets $P-M$ and $P'-M'$. The analytical sets M and M' may be given a simplicial structure (they are sums of oriented two-dimensional pseudo-manifolds having a finite number of isolated points in common). The famous Cremona-Clebsch theorem on the equality of the number of base points and fundamental curves in each of the planes P, P' implies the fact that the zero and two-dimensional Betti-groups of M and M' are isomorphic. Since neither M nor M' can have any torsion, the isomorphism of the one-dimensional Betti-groups of M and M' follows from the rationality of the curves of the homaloidal net and Cremona's reciprocity theorem on the multiplicities of a fundamental curve in a base point (cf. [1, no. 50]).

In this note, we propose to treat the case of a Cremona transformation between two complex projective spaces P and P' of complex dimension n, and to determine the relations between the additive homology invariants (Betti numbers and torsion coefficients) of the exceptional varieties $M \subset P$, $M' \subset P'$. In order to state our result, we introduce a new numerical character:

Let M be a (reducible) algebraic variety of dimension $n-1$ in n-dimensional complex projective space $P_{(n)}$. The greatest common divisor of the orders of the k-dimensional components of a generic intersection of M with a $(k+1)$-dimensional plane will be denoted by $\tau_{(k)}(M)$.

In our case, both M and M' are of complex dimension $n-1$, since they contain the Jacobian of the respective homaloidal webs. Our result will be the following:

THEOREM. *Let M and M' be the exceptional varieties in a Cremona transformation of dimension n. Then, in any dimension, the Betti Numbers of M equal those of M'. In all (real) even dimensions $2r$, the torsion coefficients of M equal those of M'. In (real) odd dimensions $2r-1$, the torsion coefficients of M and those of M' are equal, with the possible exception of one, if and only if $\tau_{(r)}(M) \neq \tau_{(r)}(M')$ in which case $\tau_{(r)}(M')$ appears as torsion coefficient of M and $\tau_{(r)}(M)$ as torsion coefficient of M' in dimension $2r-1$.*

COROLLARY. *Since there is no $2n-3$ dimensional torsion in M and M', $\tau_{(n-1)}(M) = \tau_{(n-1)}(M')$. By definition, $\tau_{(n-1)}(M)$ is the g.c.d. of the orders of the exceptional hypersurfaces in M.*

This is a result of O.-H. Keller [2]. The equality of the $2n-2$ dimensional Betti numbers has recently been established by B. Segre [3].

2. Since we will consider only homology with coefficients in the group Z of the integers, we will suppress the mention of the coefficient-group in the symbols of the homology groups. In the simplicial theory, the relative homology groups of a compact space P modulo a closed subspace M depend only on the difference-space $P-M$, so we denote them by $H_s(P-M)$.

From the topology of algebraic varieties in projective space we need only one well known fundamental fact: The fundamental cycle of an irreducible variety V_d of d (complex) dimensions and of order μ is homologous in P to μ times the $2d$-dimensional cycle carried by a (complex) d-dimensional plane in P.

In order to avoid ambiguities, all indices of dimension will forthwith denote *real* dimensions; complex dimensions will be indicated by indices in brackets.

Since for the complex projective space

$$H_{2r}(P) \cong Z, \qquad 0 \leq 2r \leq 2n,$$

$$H_{2r-1}(P) = 0, \qquad 0 < 2r-1 < 2n,$$

the exact sequence of homology of (P, M) decomposes into pieces:

$$0 \to H_{2r+1}(P-M) \xrightarrow{\partial_*} H_{2r}(M) \xrightarrow{i_*} Z \xrightarrow{p_*} H_{2r}(P-M) \xrightarrow{\partial_*} H_{2r-1}(M) \to 0$$

for all r. Since M (and M') may be considered as finite simplicial complexes, all homology groups in question are finitely generated and are characterized by their Betti numbers and torsion coefficients. We may represent each homology group $H_s(X)$ as the direct sum of a free group $H_s^0(X)$ of dimension $b_s(X)$ (its Betti number) and of the torsion group $T_s(X)$ (though the representation of this decomposition by means of generators is not canonical).

The homomorphic image in Z of an element of finite order in $H_{2r}(M)$ is always zero, hence

$$\begin{array}{c} T_{2r+1}(P - M) \cong \partial_* T_{2r+1}(P - M) \cong T_{2r}(M), \\ \wr\| \\ T_{2r+1}(P' - M') \cong \partial_* T_{2r+1}(P' - M') \cong T_{2r}(M'), \end{array} \tag{1}$$

which proves the equality of the torsion coefficients in even dimensions.

A section of M with a generic $[r+1]$-plane represents an algebraic $2r$-cycle in M which is therefore not homologous to 0 in M and in P. In fact, each of its components is mapped by i_* into μ-times a generator of $H_{2r}(P)$, and, since i_* is the injection homomorphism, these components generate the subspace $H_{2r}(M) - \partial_* H_{2r+1}(P - M)$ of $H_{2r}^0(M)$. It follows that

$$i_* H_{2r}(M) \cong \tau_{[r]}(M) \cdot Z. \tag{2}$$

Since M and M' are supposed to be the exceptional varieties of a Cremona transformation, the homeomorphism of $P - M$ onto $P' - M'$ induces an isomorphism

$$H_r(P - M) \cong H_r(P' - M')$$

and for the Betti numbers

$$\begin{array}{c} b_{2r+1}(P - M) = b_{2r}(M) - 1, \\ \| \\ b_{2r+1}(P' - M') = b_{2r}(M') - 1, \end{array} \tag{3}$$

which proves the equality of Betti numbers in even dimensions. Since in any case $\tau_{[r]}(M) \neq 0$, $\tau_{[r]}(M') \neq 0$,we have from the exact sequence

$$\begin{array}{c} 0 \rightarrow \overset{0}{H}_{2r}(P - M) \overset{\partial_*}{\rightarrow} \overset{0}{H}_{2r+1}(M) \rightarrow 0, \\ 0 \rightarrow H_{2r}(P' - M') \overset{\partial_*}{\rightarrow} \overset{0}{H}_{2r-1}(M') \rightarrow 0, \end{array} \tag{4}$$

which proves equality of Betti numbers in odd dimensions. Since in $H_{2r}(P - M)$:

$$p_* Z \cong Z_{\tau_{[r]}(M)} \tag{5}$$

with a corresponding isomorphism for $P' - M'$, we see from the exact sequence that the torsion coefficients of $P-M$ in the dimension $2r$ are those of M in the dimension $2r-1$, plus one coefficient equal to $\tau_{[r]}(M)$. As

$$T_{2r}(P-M) \cong T_{2r}(P'-M')$$

it follows that if $\tau_{[r]}(M)=\tau_{[r]}(M')$, then $T_{2r-1}(M)\cong T_{2r-1}(M')$, but if $\tau_{[r]}(M)\neq\tau_{[r]}(M')$, then $\tau_{[r]}(M)$ appears as a torsion coefficient in $T_{2r-1}(M')$ and $\tau_{[r]}(M')$ appears in $T_{2r-1}(M)$, all other torsion coefficients being equal. The theorem is completely proved.

References

1. L. Berzolari, *Algebraische Transformationen und Korrespondenzen*, Enc. Math. Wiss. (3) vol. 2, p. 111.

2. O.-H. Keller, *Ganze Cremona-Transformationen*, Monatsh. Math. Phys. vol. 47 (1939) pp. 299–306.

3. B. Segre, *Corrispondenze birazionali e topologia di varietà algebriche*, Ann. Mat. Pura Appl. (4) vol. 63 (1957) pp. 1–23.

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