RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON EMBEDDINGS OF SPHERES

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Imbed an $n-1$ sphere in an n sphere, and the complement is divided into two components. It seems that the closure of each of the resulting components should be a topological n-cell. This statement isn't true. The classical counterexample (in dimension 3) is the Alexander Horned Sphere.[1] It was conjectured, however, that if one restricts one's attention to some class of well-behaved imbeddings, then the statement is true. For instance, in the differentiable case, the Schoenflies Problem asks an even stronger question: Given $\phi: S^{n-1} \to E^n$, a differentiable imbedding of the $(n-1)$-sphere in Euclidean space, can one extend ϕ to a differentiable imbedding of the unit ball (of which S^{n-1} is the boundary) into Euclidean space?[2]

And, in fact, proofs exist for the usual categories of nice imbeddings: differentiable and polyhedral, in dimensions 1, 2, and 3.[3] The problem, then, is to prove this statement for arbitrary dimension N. Such a proof follows under a niceness condition which includes the condition of differentiability.[4]

Outline of proof. Let χ be the set of manifolds bounded by the $n-1$ sphere obtainable as the closure of a complement of a nice imbedding of S^{n-1} in S^n. Define a commutative semi-group structure in χ. (Really, it cannot be done, but just enough of a multiplication

[1] The classical such reference is Alexander's paper in the 1924 PNAS. For other amazing examples of bad imbeddings of 2-spheres in 3-space, there is an article by Artin and Fox in Volume 49 of the Annals of Mathematics.

[2] Results of Milnor (in the 1957 Annals) show that this is impossible as stated. That is, he obtains a diffeomorphism ϕ of S^6 onto itself that cannot be extended to a diffeomorphism of the unit ball in E^7 onto itself. Actually, it can be extended to a homeomorphism of the unit ball onto itself that is a diffeomorphism except at one point.

[3] There are proofs of this due to Alexander, also in the 1924 PNAS, and more recently, Moise, in the 1952 Annals.

[4] The fact that differentiable imbeddings are 'nice' in my sense is well-known, and fairly obvious. Whether or not my conditions of niceness subsume polyhedral imbeddings is an open question.

can be defined so that the rest of this goes through.) If $\boldsymbol{X}$, $\boldsymbol{X}^{-1}$ are obtained as the complementary manifolds of one imbedding of $S^{n-1}\to S^n$, then:

$$\boldsymbol{X}\cdot\boldsymbol{X}^{-1} = I^n.$$

Construct

$$\boldsymbol{X}^\infty \in \chi;\ \boldsymbol{X}^\infty = \boldsymbol{X}\cdot\boldsymbol{X}^{-1}\cdot\boldsymbol{X}\cdot\boldsymbol{X}^{-1}\cdot\boldsymbol{X}\cdots.$$

Then

$$\boldsymbol{X}^\infty = (\boldsymbol{X}\cdot\boldsymbol{X}^{-1})\cdot(\boldsymbol{X}\cdot\boldsymbol{X}^{-1})\cdots = I^n\cdot I^n\cdots = I^n.$$

But on the other hand,

$$\boldsymbol{X}^\infty = \boldsymbol{X}(\boldsymbol{X}^{-1}\boldsymbol{X})(\boldsymbol{X}^{-1}\boldsymbol{X})\cdots = \boldsymbol{X}\cdot I^n\cdot I^n\cdots = \boldsymbol{X}.$$

The above description of how the proof works is, as you shall see, a useful fiction.

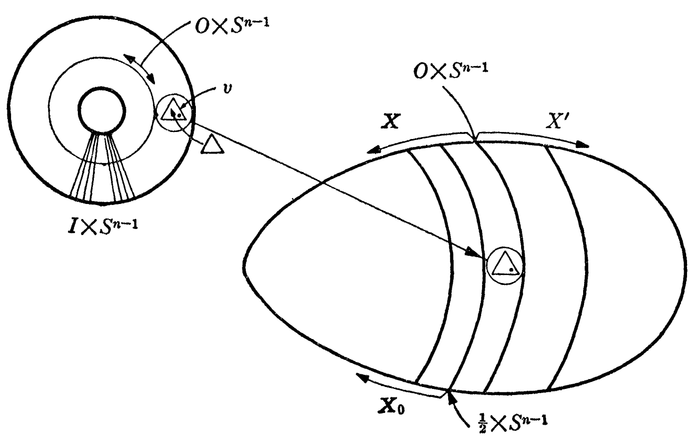

Fig. 1

The euphemism "Nice." Consider an imbedding ψ: $S^{n-1}\to S^n$ as "nice," if

(i) There is a homeomorphism ϕ: $S^{n-1}\times[-1, 1]$ into S^n such that $\phi(S^{n-1}\times 0)=\psi(S^{n-1})$.

(ii) The homeomorphism ϕ is semi-linear in the neighborhood U of some point x in $S^{n-1}\times I$. Semi-linearity is meant in regard to a subdivision of the canonical simplicial structure of S^n— the triangulation of the boundary of the $(n+1)$-simplex.

$$\boldsymbol{X}\cdot\boldsymbol{X}' = I^n:$$

Let, then, $\psi: S^{n-1} \rightarrow S^n$ be a nice imbedding. X, the set of points to the "left" of S^{n-1}, and X', the set of points to the "right" of S^{n-1} are both manifolds bounded by S^{n-1} (Figure 1). Further, since the imbedding is nice, if one calls X_0 the set of pointed to the "left" of $S^{n-1}\times 1/2$, one sees that $X \approx X_0$ (by a homeomorphism which stretches the collar of X_0 over the collar of X. Collar means the topological space $I\times S^{n-1}$). This observation says:

(1) $X+\text{Collar}=X$. (That is, if one attaches a copy of $I\times S^{n-1}$ to the boundary of X by a homeomorphism of one component of the boundary of $I\times S^{n-1}$ onto the boundary of X, the resulting manifold is homeomorphic to X.)

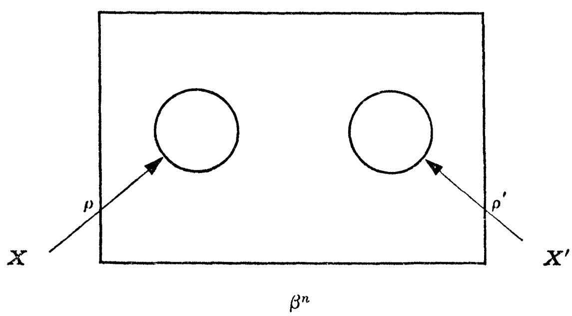

FIG. 2

Here is where I must use the second condition of niceness. Let Δ^n be a simplex in the set $U\subset I\times S^{n-1}$. Since ϕ is semilinear at that point, $\phi(\Delta^n)$ is a simplex in S^n. Call $\dot{\Delta}^n$ the interior of Δ^n, and $\beta^n = I\times S^{n-1} - \dot{\Delta}^n$. β^n shall be called an n-stock. It can be redrawn as in Figure 2. The boundary of Δ^n corresponds to the external boundary of the drawing. The name n-stock is suggested by its similarity (in dimension 2) to an obsolescent New England penal apparatus. Then ϕ restricts to a homeomorphism of β^n into $S^n - \phi(\dot{\Delta}^n)$. The space $S^n - \phi(\dot{\Delta}^n)$ is homeomorphic with I^n, being nothing more than a sphere with the interior of a simplex removed. Thus, in the light of the redrawing, Figure 2,

(2) A description of I^n may be obtained as follows: Take β^n, and sew X into one interior boundary component of β^n (by an attaching homeomorphism ρ of its boundary). Similarly, sew X' into the other interior boundary of β^n (and again by a particular homeomorphism, ρ').

X^∞:

The scene now changes. We are in Euclidean space, and have constructed something like Σ_0 of Figure 3. To construct Σ_0 start with a sequence Σ of cells laid end to end, and converging to a point. Then hollow out each chamber of Σ by extracting the interior of an n-cell, similar in shape to the chamber itself.

Any two adjacent sections form a stock, and the stocks are labelled β_i, β_i' as in Figure 3. Notice that β_i and β_i' have ω_{2i} the boundary of

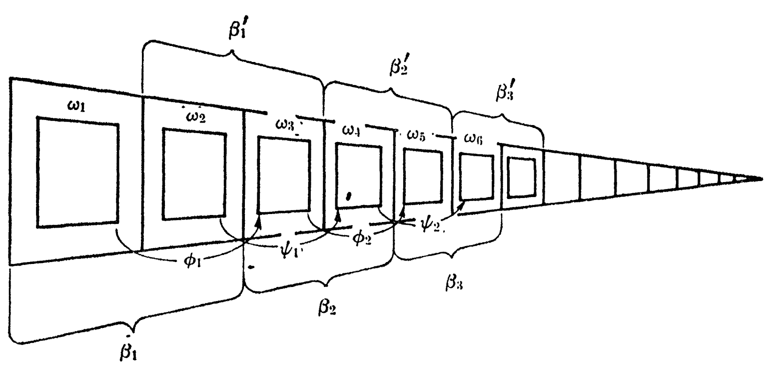

FIG. 3

an interior hole in common. Let $\zeta_i=\beta_i\to\beta_i'$ be a sequence of homeomorphisms leaving ω_{2i} pointwise invariant. Similarly, β_i' and β_{i+1} have a boundary of an interior hole ω_{2i+1} in common, and another sequence of homeomorphisms ξ_i: $\beta_i'\to\beta_{i+1}$ can be found leaving ω_{2i+2} pointwise invariant. Let $\phi_i=\zeta_i|\omega_{2i-1}$ and $\psi_i=\xi_i|\omega_{2i}$.

The object is to form X^∞ by attaching copies of X and X' in alternating holes, in such a manner that the ζ_i's extend to yield homeomorphisms of the filled-in β_i (denoted $\bar{\beta}_i$) with the filled-in β_i' (denoted $\bar{\beta}_i'$), and the ξ_i's extend similarly. Let β_1 be identified with the standard n-stock β^n. Fill up β_1 by sewing X into the first hole by ρ, (the homeomorphism of X with ω_1), and X' into the second by ρ'.

The procedure in general: to sew X into ω_{2k+1}, use the attaching homeomorphism

$$\phi_k\cdot\phi_{k-1}\cdot\cdots\cdot\phi_1\cdot\rho\colon \dot{X}\to\omega_{2k+1}.$$

Then notice that ζ_k is the identity on ω_{2k} to obtain a homeomorphism $\bar{\zeta}_k$: $\bar{\beta}_k\to\bar{\beta}_k'$ extending ζ_k to the filled-up β_k. And, in perfect analogy, to sew X' into ω_{2k+2} use the attaching homeomorphism

$$\psi_k\cdot\psi_{k-1}\cdot\cdots\cdot\psi_1\cdot\rho'\colon \dot{X}'\to\omega_{2k+2}.$$

The fact that ξ_k is the identity on ω_{2k+1} enables one to extend to a homeomorphism $\xi_k: \bar{\beta}_k' \to \bar{\beta}_{k+1}$. Since $\bar{\beta}_1 = I^n$ and $\bar{\beta}_i \approx \bar{\beta}_i' \approx \bar{\beta}_{i+1}$ it follows that $\bar{\beta}_i = \bar{\beta}_i' = I^n$ for all i.

There are two ways to view X^∞:

$$I^n = X^\infty.$$

As in figure 4.

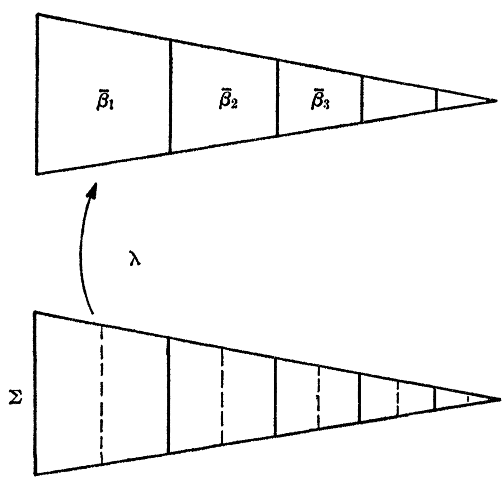

FIG. 4

The $(n-1)$-subcomplex V, of Σ consisting of the sum of the boundaries of the β_i's, is mapped by the identity homeomorphism λ into the $(n-1)$-skeleton of X^∞.

Since Σ is homeomorphic with I^n, to show that $X^\infty = I^n$, I need only show that λ can be extended to a homeomorphism $\bar{\lambda}: \Sigma \to X^\infty$.

The extension must be made to the interior of each chamber β_i, λ being already defined on the boundary.

Knowledge that $\bar{\beta}_i$ is, in fact, an n-cell reduces this to the following task:

Given a homeomorphism $\lambda: \dot{E}_1^n \to \dot{E}_2^n$ from the boundary of one n-cell to that of another, to extend it to $\bar{\lambda}: E_1^n \to E_2^n$, a homeomorphism between the two cells.

But to do this, just consider each n-cell as the unit ball in Euclidean space. Then obtain the extension homeomorphism by radial projection.

$$X^\infty = X + \text{Collar}.$$

As in figure 5.

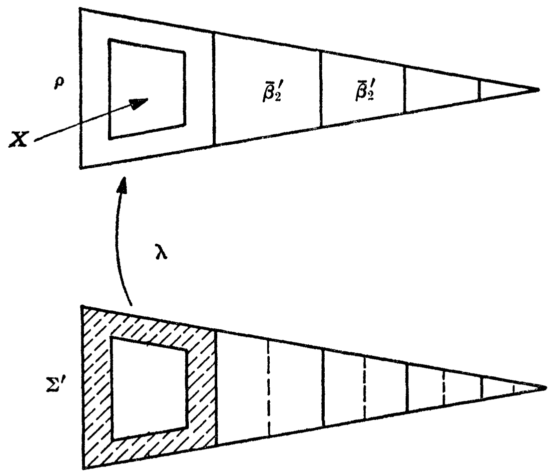

FIG. 5

All I need demonstrate is that everything but X in the above picture, i.e. $(X^\infty - X)$, is homeomorphic with $I \times S^{n-1}$. Let Σ' be Σ with the first hole whittled out, as in Figure 5. Then $\Sigma' = I \times S^{n-1}$. Moreover, Σ' is homeomorphic with $X^\infty - X$, in that the dark portion of Σ' (the sum of the first chamber and the boundaries of the $\bar{\beta}_i'$'s) is mapped by the identity homeomorphism λ onto the corresponding portion of $X^\infty - X$.

The problem then, remains to extend λ to a homeomorphism $\bar{\lambda}$ of all of Σ' onto $X^\infty - X$. This can be done by extending λ to a homeomorphism on the interior of each $\bar{\beta}_i'$. Knowledge that $\bar{\beta}_i$ is, in fact, an n-cell reduces this to the same task as above. Extension, therefore is possible.

And so:

$$I^n = X^\infty = X + \text{Collar} = X$$

which proves that:

If S^{n-1} is embedded nicely in S^n, the closure of each of the complementary components is topologically an n-cell.

Some open problems. The second condition of "niceness" is, as far as I can see, merely a technical contrivance needed to make this method of proof work. It would be very odd if it really mattered. Yet no simple rephrasing of the proof enables one to get around some such restriction—at least I have found none such. Further, from the point of view of certain applications of the theorem, this restriction is extremely unpleasant. For example, if one had the unrestricted theorem, it would be possible to prove:

(1) If the open cone of a topological space X is locally euclidean at the origin, then it is topologically equivalent with euclidean space. (The open cone of a space is obtained by taking $X \times R^*$, where R^* is the topological space consisting of the non-negative real numbers, and identifying $X \times 0$ to a point. The origin is just the image of $X \times 0$.) As it stands, one can prove the above where X is a finite complex.

(2) Let K be a simplicial complex topologically equal to the n-sphere. Let D^n be an n-simplex in K. Find an n-simplex D_0^n contained in the interior of D^n, and similar to it. Then $\dot{D}_0^n$ is an $n-1$ sphere embedded in S^n with a product neighborhood, and bounds the two regions D_0^n and its complement. The complement of the interior of D_0^n is topologically an n-cell.

(3) Addition of topological manifolds is a well-defined operation. Thus, there is the problem of determining whether condition (ii) of the definition of niceness can be eliminated.

Also, there is the question of polyhedral imbeddings. Is every polyhedral imbedding "nice" in the sense of the above definition? Here the difficulty is with condition (i). There is a lemma of Noguchi which is as yet unpublished which shows that it is so for combinatorial imbeddings up to dimension $n=5$.

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