Cardinal and ordinal numbers. By W. Sierpiński. Monografie Matematyczne, vol. 34. Warszawa, Panstwowe Wydawnictwo Naukowe, 1958. 487 pp.

Not since the publication in 1928 of his *Leçons sur les nombres transfinis* has Sierpiński written a book on transfinite numbers. The present book, embodying the fruits of a lifetime of research and experience in teaching the subject, is therefore most welcome. Although generally similar in outline to the earlier work, it is an entirely new book, and more than twice as long. The exposition is leisurely and thickly interspersed with illuminating discussion and examples. The result is a book which is highly instructive and eminently readable. Whether one takes the chapters in order or dips in at random he is almost sure to find something interesting. Many examples and applications are included in the form of exercises, nearly all accompanied by solutions.

The exposition is from the standpoint of naive set theory. No axioms, other than the axiom of choice, are ever stated explicitly, although Zermelo's system is occasionally referred to. But the role of the axiom of choice is a central theme throughout the book. For a student who wishes to learn just when and how this axiom is needed this is the best book yet written. There is an excellent chapter devoted to theorems equivalent to the axiom of choice. These include not only well-ordering, trichotomy, and Zorn's principle, but also several less familiar propositions: Lindenbaum's theorem that of any two nonempty sets one is equivalent to a partition of the other; Vaught's theorem that every family of nonempty sets contains a maximal disjoint family; Tarski's theorem that every cardinal has a successor, and other propositions of cardinal arithmetic; Kurepa's theorem that the proposition that every partially ordered set contains a maximal family of incomparable elements is an equivalent when joined with the ordering principle, i.e., the proposition that every set can be ordered. It is shown that the ordering principle can be deduced from the existence for any set of a function associating with each subset having at least two elements one of its non empty proper subsets. Included also is the author's deduction of the axiom of choice from the generalized continuum hypothesis, and Mostowski's theorems concerning the restriction of the axiom of choice to families of sets of n elements. However, topological equivalents such as Tychonoff's theorem are not discussed.

The first five chapters deal with sets and elementary operations, equivalence and comparison of sets, denumerable sets, and sets of power $\mathfrak{c}$. The axiom of choice is introduced in Chapter 6, but the discussion of equivalents is deferred until later. In the early chapters the author stresses a metamathematical distinction between sets and mappings that are defined "effectively" and those whose mere "existence" is asserted. An intuitive sense for this distinction is conveyed by discussion and example, but some readers may wish that this notion could have been defined more precisely.

The term "cardinal number" is used synonymously with "power." The theory of cardinal numbers is presented in some 50 pages, followed by 180 pages devoted to ordered sets, order types, and a long chapter on ordinal numbers. Next comes a chapter on the alephs, and two final chapters on equivalents and applications of the axiom of choice. One rather unexpected inversion results from presenting cardinal before ordinal theory. To define the sum of an infinite series of cardinals one needs a sequence of sets having the given powers. It is not obvious how these can be chosen, even with the aid of the axiom of choice, since the classes of all sets having the given powers is not a family of sets to which the axiom applies. The author resolves this difficulty only after proving the well-ordering theorem and introducing ordinal numbers. Then it becomes possible to associate with each cardinal the set of ordinals that precede the corresponding initial ordinal.

It is stated in the foreword that the Polish manuscript of the book was completed in 1952, and that it was impossible to include many later results, in particular those contained in Bachmann's *Transfinite Zahlen*. For instance, the sharper estimates of Hartog's function $\aleph(m)$ obtained by Specker in 1954, which make possible an improvement in the deduction of the axiom of choice from the generalized continuum hypothesis, are not included. However, such omissions are rare, and the expert as well as the student will find the book very useful.

To indicate something of the flavor of the book, here are a few typical items selected rather arbitrarily, most from among the exercises. (A) denotes the axiom of choice, and n and m denote infinite cardinals.

Without (A) one cannot prove that every infinite set is equivalent to a proper subset, but one can prove this for its second power set (p. 115). Hence (p. 141) $n=n+1$ for any n of the form $n=2^{2^m}$. It follows that $n^2=n$ for any n of the form $n=2^{2^k}$ where $k=2^{2^m}$. (It is well known that "$n^2=n$ for all n" is equivalent to (A).)

We can define effectively a correspondence associating with each set of at least two real numbers a nonempty proper subset (p. 106). It follows that $m \leqq 2^{\aleph_0}$ implies $2^{\aleph_0} \leqq 2^m$ (p. 148).

Without (A), $m^2 > \aleph_0$ implies $m > \aleph_0$. But without (A) we cannot prove that $m^2 > n^2$ either implies or is implied by $m > n$ (p. 149).

Without (A) we can prove that the set of all infinite sequences of real numbers is of power $\mathfrak{c}$, but without (A) we are unable to prove that the set of all denumerable sets of real numbers is of power $\mathfrak{c}$ (p. 112).

The order types $\alpha = \omega\eta$ and $\beta = \omega(\eta + 1)$ satisfy $\alpha^2 = \beta^2$ and $\alpha \neq \beta$. But it is an open question whether there exist order types γ and δ for which $\gamma^3 = \delta^3$ and $\gamma^2 \neq \delta^2$ (p. 232).

Fermat's last theorem is false for order types (p. 232), and for ordinals with transfinite exponent (p. 318). Fermat numbers, i.e. ordinals of the form $2^{2^\alpha} + 1$, are prime for every transfinite ordinal α (p. 339).

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Gitterpunkte in mehrdimensionalen Kugeln. By A. Walfisz. Monografie Matematyczne, vol. 33. Warszawa, Panstwowe Wydawnictwo Naukowe, 1957. 471 pp.

This is a beautifully written book by a leading expert in the field. Although of immense value to the specialist, it is addressed to a wider circle of readers. To quote the author's own words, "Das Studium des Buches setzt nur Kenntnisse voraus, wie sie in den üblichen Anfängervorlesungen über Analysis, Algebra und Zahlentheorie an den Hochschulen gegeben werden. Auch sind die Rechnungen über all sehr eingehend durchgeführt." Almost a third of the book is devoted to researches of the last ten years.

The book is concerned with the study of $P_k(X)$, which is the difference between the number of lattice-points in the k-dimensional hypersphere

$$y_1^2 + y_2^2 + \cdots + y_k^2 \leqq X \tag{1}$$

and its volume $V_k(X)$. So

$$P_k(X) = A_k(X) - V_k(X) \tag{2}$$

where $A_k(X)$ is the number of lattice-points satisfying (1). It is well-known that we have the asymptotic relation $A_k(X) \sim V_k(X)$.

Gauss observed that $P_2(X) = O(X)^{1/2}$. Sierpinski (1909) found $P_2(X) = O(X^{1/3})$, van der Corput proved the sharper result $P_2(X) = O(X^c)$ and $c < 1/3$ and there have been petty improvements in the exponent in later years. It is also known (Hardy) that the exponent cannot be lowered below $1/4$; on the other hand it is considered highly