RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

A TOPOLOGICAL PROOF OF THE CONTINUITY OF THE DERIVATIVE OF A FUNCTION OF A COMPLEX VARIABLE

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In this paper the continuity of the derivative of an analytic function of a complex variable is proved in an elementary, or purely topological, fashion. That is, no use whatever is made of complex integration or equivalent tools. The desirability of such a proof has been emphasized in *Complex analysis* by L. V. Ahlfors [**1**, p. 82], and even more recently in *Topological analysis* by G. T. Whyburn [**2**, p. 89]. Our proof has been made accessible only by the extensive modern development of the subject of topological analysis (see [**2**] for rationale and bibliography). The author wishes to express his appreciation to Professor G. T. Whyburn for suggesting the feasibility of attacking this problem at this time.

Throughout, we shall be concerned with a nonconstant complex valued function $f(z)$ defined and having a finite derivative at each point of an open connected set E of the complex plane. We shall employ Theorems A and B in the proof of the main theorem.

THEOREM A. *A necessary and sufficient condition that f be a local homeomorphism at $z_0 \in E$ is that $f'(z_0)$ be not zero* [**2**, p. 85].

THEOREM B. *If A and B are 2-manifolds without edges and $f(A) = B$ is a light open mapping, then for any $y \in B$ and $x \in f^{-1}(y)$, there exist 2-cell neighborhoods U of x and V of y such that $f(U) = V$ and the mapping f of U onto V is topologically equivalent to a power mapping $w = z^k$ on $|z| \leqq 1$, for some positive integer k* [**2**, p. 88].

We shall also use Rouché's theorem [**2**, p. 93], and the lemmas which follow the next definition.

DEFINITION. *Let $\{z_i\}$ be a sequence of points of E. A sequence $\{\bar{z}_i\}$ of*

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points of E is called an f-companion sequence for $\{z_i\}$ if and only if there exists a positive integer I such that, for each $i>I$, $\bar{z}_i \neq z_i$ but $f(\bar{z}_i)=f(z_i)$.

Note that it is clear from Theorem A that $f'(z_0)=0$ if and only if there exists a pair of f-companion sequences converging to z_0. However, we will need a slightly stronger condition than this.

LEMMA 1. *If $f'(z_0)=0$ and if U and V are 2-cell neighborhoods (as in Theorem* B) *of z_0 and $f(z_0)$, respectively, such that $f(U)=V$ and the mapping f: U onto V is topologically equivalent to the mapping $w=z^k$ on $|z| \leqq 1$ for some positive integer k, then $k>1$.*

PROOF. If $k=1$, then the mapping f: U onto V is topologically equivalent to the identity mapping of $|z| \leqq 1$. Hence this mapping is a homeomorphism and f is a local homeomorphism at z_0 (noting, from the proof of Theorem B as given in [2], that z_0 is an *interior* point of U). Thus, by Theorem A, $f'(z_0) \neq 0$.

LEMMA 2. *A necessary and sufficient condition that $f'(z_0)=0$ is that, for each infinite sequence $\{z_i\}$ in $E-\{z_0\}$ converging to z_0, there exist an f-companion sequence $\{\bar{z}_i\}$ in $E-\{z_0\}$ converging to z_0.*

PROOF. *Sufficiency.* The existence of a pair of f-companion sequences converging to z_0 precludes the possibility that f is a local homeomorphism at z_0. Therefore, $f'(z_0)$ must be zero, by Theorem A.

Necessity. Let D be an open 2-cell neighborhood of z_0 with $\bar{D} \subset E$ and such that $f(z) \neq f(z_0)$ for any $z \in \bar{D}-\{z_0\}$. The existence of such a D results from the "scattered inverse property" [2, p. 83]. The mapping $f|D$ satisfies the hypothesis of Theorem B, so there exist 2-cell neighborhoods $U \subset D$ and $V \subset f(D)$ of z_0 and $f(z_0)$, respectively, such that $f(U)=V$ and the mapping f: U onto V is topologically equivalent to the mapping $w=z^k$ on $|z| \leqq 1$, for some positive integer k. By Lemma 1, $k>1$, since $f'(z_0)=0$. Let g denote the homeomorphism of U onto $|z| \leqq 1$ and h the homeomorphism of $|w| \leqq 1$ onto V such that $f(z)=h([g(z)]^k)$, for all $z \in U$. Consider now $\{g(z_i)\}$. Since for at most a finite number of subscripts can $g(z_i)=0$, we assume that this happens for none. There exists an I such that, for all $i>I$, $z_i \in U$ and, for each of these, there exists a $p_i \neq g(z_i)$ with $|p_i| \leqq 1$ and $p_i^k=[g(z_i)]^k$. Let $\bar{z}_i=g^{-1}(p_i)$, for each $i>I$, and for $i=1, \cdots, I$ define $\bar{z}_i$ arbitrarily in $D-\{z_0\}$. Then clearly $\{\bar{z}_i\}$ is an f-companion sequence for $\{z_i\}$. That $\bar{z}_i=z_0$ for no i, and that $\bar{z}_i \to z_0$ are consequences of the fact that for no z in $\bar{D}-\{z_0\}$ is $f(z)=f(z_0)$.

We are now ready for the main theorem.

THEOREM C. *If $f'(z)$ exists for all z in E, then f' is continuous in E.*

PROOF. Suppose that f' is not continuous at $z_0 \in E$. We shall assume (without loss of generality) that $f'(z_0)=0$. (Otherwise, replace $f(z)$ by $g(z)=f(z)-[f'(z_0)]z$ and note that $g'(z)=f'(z)-f'(z_0)$ is continuous if and only if $f'(z)$ is.) The assumption that f' is not continuous at z_0 implies that there exist an $\epsilon>0$ and a sequence $\{z_i\}$ in E converging to z_0 such that $|f'(z_i)| \geqq \epsilon$, for all i. By Lemma 2, there exists for $\{z_i\}$ an f-companion sequence $\{\bar{z}_i\}$ in E converging to z_0.

For each i and for $z \in E$, define

$$N_i(z) = \begin{cases} \dfrac{f(z) - f(z_i)}{z - z_i}, & \text{if } z \neq z_i \\ f'(z_i), & \text{if } z = z_i. \end{cases}$$

Let $\{E_i\}$ be a sequence of open 2-cell neighborhoods of z_0, each contained in E and such that diam $E_i \to 0$, as $i \to \infty$. We may assume that each E_i contains both z_i and $\bar{z}_i$, since this can be arranged by taking subsequences. Now on E_i, $N_i(z)$ is continuous and, if I is the positive integer such that, for $i>I$, $\bar{z}_i \neq z_i$ but $f(\bar{z}_i)=f(z_i)$, then, for all $i>I$, the connected set $N_i(E_i)$ contains both zero and a point whose modulus is greater than or equal to ϵ. Hence, for $i>I$, E_i contains a point z_i^* such that $z_i^* \neq z_i$ and $N_i(z_i^*)=t_i$ where $|t_i|=\epsilon/2$. Also, the sequence $\{z_i^*; i>I\}$ converges to z_0.

Suppose that $\{t_i;\ i>I\}$ is an infinite sequence. Then, since all t_i belong to the compact set $|z|=\epsilon/2$, there exists a limit point t such that $|t|=\epsilon/2$ and (again by taking subsequences back down the line) we may suppose that $t_i \to t$. Let $g(z)=f(z)-tz$. Then $g'(z_0)=-t\neq 0$, so g is a local homeomorphism at z_0. Suppose D, with boundary C, is an open circular neighborhood of z_0, centered at z_0, with $D\cup C \subset E$, and such that g is a homeomorphism on $D \cup C$. Let $F(z)=g(z)-g(z_0)$, $F_i(z)=g(z)-g(z_i)$, and $G_i(z)=(t-t_i)(z-z_i)$. Let δ denote the minimum of $\{|F(z)| : z \in C\}$. Since $0 \notin F(C)$, we have $\delta>0$. Since F_i converges uniformly to F on C, there exists a positive integer I_1 such that $i>I_1$ implies that the minimum of $\{|F_i(z)| : z \in C\} > 2\delta/3$. Since $t_i \to t$ and $z_i \to z_0$, there exists an I_2 such that, for all $i>I_2$ and for all $z \in C$, $|G_i(z)|<\delta/3$. Finally, there is an I_3 such that, for all $i>I_3$, both z_i and z_i^* belong to D.

Fix i at a value greater than $I+I_1+I_2+I_3$. Then $|G_i(z)|$ is less than $|F_i(z)|$ for all $z \in C$. By Rouché's theorem, $F_i(z)+G_i(z)$ has exactly the same number of zeros in D as $F_i(z)$, which has only one since g is a homeomorphism on D. But

$$\begin{aligned} F_i(z) + G_i(z) &= g(z) - g(z_i) + (t - t_i)(z - z_i) \\ &= [g(z) + (t - t_i)z] - [g(z_i) + (t - t_i)z_i] \\ &= [f(z) - t_i z] - [f(z_i) - t_i z_i] \\ &= f(z) - f(z_i) - t_i(z - z_i) \\ &= [N_i(z) - t_i](z - z_i), \end{aligned}$$

which has a zero when $z = z_i$ and also when $z = z_i^*$. This contradiction proves the theorem in case $\{t_i;\ i > I\}$ is an infinite sequence.

In case $\{t_i;\ i > I\}$ is finite, we obtain an even more immediate contradiction.

References

1. L. V. Ahlfors, *Complex analysis*, New York, 1953.
2. G. T. Whyburn, *Topological analysis*, Princeton, 1958.

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