RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

SPLINE FUNCTIONS, CONVEX CURVES AND MECHANICAL QUADRATURE[1]

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Communicated July 10, 1958

The following lines describe some closely related results concerning the three subjects of the title. Detailed proofs will be given elsewhere.

1. **Spline functions.** Let x_+^{n-1} denote the truncated power function defined as x^{n-1} if $x \geqq 0$ and $=0$ if $x < 0$ $(n = 1, 2, \cdots)$. Let ξ_ν $(\nu=1, \cdots, k)$ be a given finite sequence of increasing abscissae. By a *spline function* of degree $n-1$ we mean a function of the form

$$S_{n-1,k}(x) = P_{n-1}(x) + \sum_{\nu=1}^{k} C_\nu(x - \xi_\nu)_+^{n-1}, \tag{1}$$

where $P_{n-1}(x)$ is a polynomial of degree $\leqq n-1$. Equivalently, this function may be defined by separate polynomials of degree $\leqq n-1$ in each of the $k+1$ intervals $(-\infty, \xi_1)$, (ξ_1, ξ_2), $\cdots$, (ξ_k, ∞), such that the composite function has $n-2$ continuous derivatives for all real x. For $n=1$ we obtain a step-function, for $n=2$ a continuous broken-line graph and so on. The ξ_ν are called the ***knots*** of the spline function. The reasons for the name "spline function" are explained in [**5**, p. 67].

By adding to the spline (1) the monomial x^n we obtain a function

$$F(x) = x^n + S_{n-1,k}(x) \tag{2}$$

which we call a *monospline* of degree n and knots ξ_ν. Both splines and monosplines become polynomials if $k=0$. Much of the familiar Algebra of polynomials disappears if $k>0$, as these systems are not closed with respect to multiplication. Fortunately much of the Calculus of polynomials survives such as the relations

[1] This paper was prepared partly under the sponsorship of the United States Air Force, Office of Scientific Research, ARDC, under a contract with the University of Pennsylvania.

$$\frac{d}{dx}S_{n-1,k}(x) = S_{n-2,k}(x), \qquad \int S_{n-1,k}(x)dx = S_{n,k}(x),$$

with similar ones for monosplines. The periodic extension of period unity of the Bernoulli polynomial $B_n(x)$, $(0 \leqq x \leqq 1)$, is a monospline of degree n having knots in all points of integral abscissae, an example which is familiar from the theory of the Euler-Maclaurin sum formula.

We now procede to show that the so-called fundamental theorem of Algebra also holds for monosplines. However, due to the nature of these functions we must restrict ourselves exclusively to the *real field.* We begin by defining the *multiplicity* or *order* of a zero of (2).

This notion being evident if the zero is not a knot of (2), we may assume that it is one, $x = \xi$ say. Even now the meaning of a zero $x = \xi$ of multiplicity $< n-2$ is the usual one and we may therefore restrict our discussion to the case when

$$F(\xi) = F'(\xi) = \cdots = F^{(n-2)}(\xi) = 0.$$

But then the function (2) has the form

$$F(x) = \begin{cases} A(x-\xi)^{n-1} + (x-\xi)^n & \text{if } x < \xi, \\ B(x-\xi)^{n-1} + (x-\xi)^n & \text{if } x > \xi, (A \neq B), \end{cases}$$

a representation which is valid in a neighborhood of ξ whose end-points are the nearest knots.

We may now define the order of the zero $x = \xi$ as follows:

DEFINITION. 1. *If $AB > 0$ we say that ξ is a zero of order $n-1$.*

2. *If $AB < 0$ we say that ξ is a zero of order n.*

3. *If $AB = 0$ there are two subcases:*

3′. *If $B - A > 0$ we say that ξ is a zero of order n,*

3″. *If $B - A < 0$ we say that ξ is a zero of order $n+1$.*

The following is readily seen: 1. The largest possible multiplicity of a zero of (2) is $n+1$. 2. $F(x)$ changes sign at $x = \xi$ or not, depending on whether the order of the zero is odd or even. 3. A zero of $F(x)$ is also a zero of its derivative $F'(x)$ of order by exactly one unit less than before differentiation.

Our analogue of the fundamental theorem of Algebra is

THEOREM 1. *A monospline* (2) *can have at most $n+2k$ zeros, counting multiplicities as defined above. Given arbitrarily the zeros $x_1 < x_2 < \cdots < x_s$ with corresponding multiplicities $\alpha_1, \alpha_2, \cdots, \alpha_s$, such that*

$$\alpha_i \leqq n + 1, \qquad (i = 1, \cdots, s), \tag{3}$$

$$\sum_{1}^{s} \alpha_i = n + 2k, \tag{4}$$

there is a uniquely defined monospline (2), *of degree* n, *and* k *distinct knots* $\xi_1, \cdots, \xi_k$, *having the given set of zeros* x_i *of orders* α_i, *respectively. Thus* $n+2k$ *plays the role of the "degree" of* (2).

If $k=0$ Theorem 1 reduces to the familiar theorem for the field of reals. With k arbitrary, the cases when $n=1$ or $n=2$ are easily established and furnish instructive examples for Theorem 1 which are readily solved graphically. The general case of Theorem 1 depends on a geometric result to which we now turn.

2. **Convex curves.** Let Γ be a compact set in the $2k$-dimensional euclidean space E_{2k} and let $K=K(\Gamma)$ denote the convex hull of Γ. A well known theorem of Carathéodory states that every point p of K may be obtained as a centroid with positive masses of $2k+1$ appropriately chosen points of Γ (see [2, pp. 35–36]). The determination of the knots $\xi_1, \cdots, \xi_k$ of a monospline (2) of given zeros (Theorem 1) does not depend on the solution of readily available algebraic equations, except if $n=1$ or 2 and other special cases to be mentioned in §3. Rather the proof of their existence depends on the following refinement of Carathéodory's theorem under very special circumstances which allow to reduce Carathéodory's number $2k+1$ to $k+1$.

We assume that Γ is a closed curve in E_{2k} given in parametric form and not contained in a hyperplane. We say that Γ is *convex in* E_{2k} provided that the curve Γ crosses no hyperplane more than $2k$ times. Let us now assume that our curve Γ is convex in E_{2k} and let p be a point in the interior of $K(\Gamma)$. Under these special assumptions the following theorem holds:

THEOREM 2. *Given at will a point* q_0 *on* Γ, *there exist* k *further points* $q_1, \cdots, q_k$, *on* Γ, *such that* p *is a centroid of the* $k+1$ *points* $q_0, q_1, \cdots, q_k$ *with positive masses.*

The case $k=1$ of convex curves in the plane E_2 is obviously true. Another case of Theorem 2 familiar from the theory of the trigonometric moment problem is obtained if Γ is the special curve

$$\begin{aligned} &x_1 = \cos t, \quad && x_3 = \cos 2t, \cdots, x_{2k-1} = \cos kt, \\ &x_2 = \sin t, \quad && x_4 = \sin 2t, \cdots, x_{2k} = \sin kt, \qquad (0 \leqq t \leqq 2\pi). \end{aligned}$$

The proof of the general case depends on results obtained in [6, §1], and in a joint paper [1] with H. B. Curry.

3. Mechanical quadrature. It should come as no surprise that monosplines are related to the problem of mechanical quadrature, for indeed the kernels which appear in Peano's form of their remainders are precisely monosplines (see [**3**]). This connection furnishes the occasion to mention two further examples illustrating Theorem 1.

EXAMPLE 1. Let $n=2k$. According to Theorem 1 a monospline

$$F(x) = x^{2k} + S_{2k-1,k}(x) \tag{5}$$

is uniquely defined if we preassign its $n+2k=4k$ zeros, none of multiplicity exceeding $2k+1$. We now prescribe these zeros to be $x=-1$ and $x=+1$, each of multiplicity $2k$. The resulting monospline (5) is easily found to be identical with the Peano-kernel in Gauss' formula of mechanical quadrature. In fact the knots $\xi_1, \cdots, \xi_k$ of (5) are the zeros of the kth Legendre polynomial.

EXAMPLE 2. Let again $n=2k$ and consider

$$F(x) = x^{2k} + S_{2k-1,k+1}(x) \tag{6}$$

which may have as many as $2k+2(k+1)=4k+2$ zeros. We now prescribe these zeros to be $x=-1$ and $x=+1$, each of multiplicity $2k+1$ (notice that the requirements (3), (4), of Theorem 1 are verified). The resulting monospline (6) is found to be identical with the Peano-kernel in Radau's formula of mechanical quadrature. (See [**4**, formula (24 bis), p. 296]; also [**7**, p. 161, Example 2]). The knots of (6) are, besides ± 1, the zeros of the *derivative* of the kth Legendre polynomial.

A more general application of Theorem 1 yields the following two theorems:

THEOREM 3. *We are given two integers k and n such that*

$$1 \leqq n \leqq 2k \tag{7}$$

and we set

$$r = 2k - n. \tag{8}$$

We are also given r abscissae x_i such that

$$-1 < x_1 < x_2 < \cdots < x_r < +1, \tag{9}$$

where the set $\{x_i\}$ is void if $r=0$.

There exists a uniquely defined quadrature formula

$$\sum_{\nu=1}^{k} G_\nu f(\xi_\nu) = \int_{-1}^{1} f(x)dx, \tag{10}$$

where

$$(11) \qquad -1 < \xi_1 < \cdots < \xi_k < 1, \qquad G_1 > 0, \cdots, G_k > 0,$$

and such that the relation (10) *holds for every spline function of the form*

$$(12) \qquad f(x) = P_{n-1}(x) + \sum_{i=1}^{r} C_i(x - x_i)_+^{n-1},$$

i.e. of degree $n-1$ *and having the* r *preassigned knots* (9).

We wish to call (10) a quadrature formula of the *Gaussian type* for the following two reasons: First, if $n=2k$ hence $r=0$, then (10) reduces to Gauss' formula, as it must, since (12) now reduces to an arbitrary polynomial of degree $2k-1$. Secondly, notice that in the general case the spline function (12) depends on $n+r=n+(2k-n)=2k$ arbitrary parameters. In other words, we get a k-point formula (10) enjoying the characteristic "double precision" for a preassigned set of $2k$ functions

$$1, x, x^2, \cdots, x^{n-1}, (x - x_i)_+^{n-1} \qquad (i = 1, \cdots, 2k - n).$$

THEOREM 4. *With assumptions* (7), (8), (9), *identical with those of Theorem* 3, *there exists a uniquely defined quadrature formula*

$$(13) \qquad \sum_{\nu=0}^{k} R_\nu f(\zeta_\nu) = \int_{-1}^{1} f(x)dx,$$

where

$$(14)\ \zeta_0 = -1 < \zeta_1 < \cdots < \zeta_{k-1} < \zeta_k = +1, \ R_0 > 0, \cdots, R_k > 0,$$

and such that the relation (13) *holds for every spline function of the form* (12).

We wish to call (13) a quadrature formula of the *Radau type*. Indeed, if $n=2k$ hence $r=0$, then (13) reduces to Radau's formula (loc. cit.). Also the "double precision" argument applies as before.

The practical implications of Theorems 3 and 4 are as follows: The accuracy of the formulae of Gauss and Radau is well known (See Radau's paper [**4**, pp. 334–335] for an instructive series of numerical examples). A serious drawback for the practical computer is the irrationality of the knots $\xi_\nu(\zeta_\nu)$ and weights $G_\nu(R_\nu)$. Theorems 3 and 4 allow to construct quadrature formulae having *simple rational* knots and weights and sharing the accuracy of the Gauss and Radau formulae for *any preassigned degree of exactness* n *in the range* $1 \leqq n < 2k$. Some such formulae are already in common use. For instance, the formula of G. F. Hardy (see [7, p. 151]) is a Radau type formula with $k=4$ and $n=6$, a fact which I may be permitted to interpret as explaining its good accuracy.

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