ON RAPIDLY MIXING TRANSFORMATIONS AND AN APPLICATION TO CONTINUED FRACTIONS

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1. Let Ω be a measure space (of total measure 1) and let T be a measure preserving transformation in the sense that

$$\mu\{T^{-1}(A)\} = \mu\{A\}, \tag{1.1}$$

where $T^{-1}(A)$ denotes the inverse image of A (we do not assume that T is necessarily one-to-one).

Let

$$V(\omega) = \begin{cases} 1, & \omega \in B, \\ 0, & \omega \notin B, \end{cases} \tag{1.2}$$

and consider

$$\sum_{k=1}^{n} V(T^k\omega). \tag{1.3}$$

It will be our purpose to sketch a proof of the theorem that under a suitable condition on T the "conditional" measure

$$\frac{\mu\left\{\dfrac{\sum_{k=1}^{n} V(T^k\omega)}{n} - \mu\{B\} < \dfrac{\alpha}{n^{1/2}},\ \omega \in B\right\}}{\mu\{B\}} \tag{1.4}$$

approaches as n tends to infinity

$$\frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^{\alpha} e^{-u^2/(2\sigma^2)} du, \tag{1.5}$$

where σ is, in general, not known explicitly.

The condition to be imposed on T is that of "exponentially rapid mixing" and can be stated as follows:

If ν and ν_m are defined by

$$\nu\{A\} = \frac{\mu\{A \cap B\}}{\mu\{B\}}, \qquad \nu_m\{A\} = \frac{\mu\{A \cap B \cap T^{-m}B\}}{\mu\{B \cap T^{-m}B\}}$$

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then for every measurable set A

$$
(1.6)\quad \begin{aligned} &|\nu\{T^{-k}A\} - \mu\{T^{-k}A\}| = |\nu\{T^{-k}A\} - \mu\{A\}| \leqq He^{-\epsilon k}\mu\{A\},\\ &|\nu_m\{T^{-k}A\} - \mu\{A\}| \leqq H_m e^{-\epsilon k}\mu\{A\}. \end{aligned}
$$

$\epsilon > 0$ is an absolute constant, but H_m may vary with m.

This condition may seem more severe than Bernstein's (Math. Ann. vol. 97 (1927) pp. 1–59), but Bernstein's conditions would ask for a "uniformly rapid" decrease of all expressions

$$
\left| \frac{\mu\{T^{-k}A \cap B_1 \cap \cdots \cap B_m\}}{\mu\{B_1 \cap \cdots \cap B_m\}} - \mu\{A\} \right|
$$

where each $B_i = B$ or $\Omega - B$.

2. Let ν be defined as follows:

$$
(2.1)\qquad \nu(A) = \frac{\mu(A \cap B)}{\mu(B)},
$$

where B is a fixed set of positive measure. We set

$$
(2.2)\qquad {}_{(k)}P_n^{(1)} = \frac{\int_\Omega V(T^k\omega) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_\Omega V(T^k\omega) d\nu},
$$

$$
(2.3)\qquad {}_{(k)}P_n^{(2)} = \frac{\int_\Omega (1 - V(T^k\omega)) \exp\left(-u \sum_{r=k+1}^{k+n} V(T^r\omega)\right) d\nu}{\int_\Omega (1 - V(T^k\omega)) d\nu}
$$

(with the obvious convention $P_0 \equiv 1$) and

$$
(2.4)\qquad S_k^{(q)} = \sum_{n=0}^{\infty} {}_{(k)}P_n^{(q)} z^n, \qquad q = 1, 2.
$$

Setting also

$$
B^{(1)} = B, \qquad B^{(2)} = \Omega - B \qquad V^{(1)} = 1, \qquad V^{(2)} = 0
$$

we verify by an immediate calculation that

$$
(2.5)\qquad \sum_q \nu\{T^{-k}(B^{(q)})\} S_k^{(q)} = 1 + z \sum_q e^{-uV^{(q)}} \nu\{T^{-(k+1)}(B^{(q)})\} S_{k+1}^{(q)},
$$

where $\nu\{T^{-k}(B^{(q)})\}S_k^{(q)}$ is taken to be zero whenever $\nu\{T^{-k}(B^{(q)})\}=0$.

Multiplying both sides of (2.5) by z^k and summing on k from 0 to ∞ we obtain

$$S_0^{(1)} = \frac{1}{1-z} - (1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B^{(1)})\}S_k^{(1)}z^k. \tag{2.6}$$

It should, of course, be remembered that S depends on z and u. Dropping the superscript 1, we get, setting

$$Q(z) = \sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}(S_k - S_0)z^k, \tag{2.7}$$

$$\begin{aligned} S_0 &= \frac{1}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]} \\ &\quad - \frac{(1-z)(1-e^{-u})Q(z)}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]}. \end{aligned} \tag{2.8}$$

3. Let us expand $(1-e^{-u})(1-z)Q(z)$ in a power series

$$(1-e^{-u})(1-z)Q(z) = \sum_{r=1}^{\infty}\gamma_r(u)z^r \tag{3.1}$$

(recall that (2.7) is not a power series expansion of Q since the S's themselves depend on z) and let us also write

$$\frac{1}{(1-z)\left[1+(1-e^{-u})\sum_{k=1}^{\infty}\nu\{T^{-k}(B)\}z^k\right]} = \sum_{r=0}^{\infty}\beta_r(u)z^r. \tag{3.2}$$

From (2.8) we have comparing coefficients

$$\begin{aligned} {}_{(0)}P_n &= \frac{\int_B \exp\left(-u\sum_{r=1}^{n}V(T^r\omega)\right)d\nu}{\int_B d\nu} \\ &= \frac{\int_B \exp\left(-u\sum_{r=1}^{n}V(T^r\omega)\right)d\mu}{\mu\{B\}} = \beta_n(u) + \sum_{r=1}^{n}\gamma_r(u)\beta_{n-r}(u), \end{aligned} \tag{3.3}$$

and hence setting $u = -i\xi/n^{1/2}$,

$$\begin{aligned}
&\frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}}\left[\sum_1^n V(T^r\omega) - n\mu\{B\}\right]\right)d\mu}{\mu\{B\}} \\
(3.4) \qquad &= e^{-i\xi\mu\{B\}n^{1/2}}\beta_n\left(-\frac{i\xi}{n^{1/2}}\right) \\
&\quad + e^{-i\xi\mu\{B\}n^{1/2}}\sum_{r=1}^n \gamma_r\left(-\frac{i\xi}{n^{1/2}}\right)\beta_{n-r}\left(-\frac{i\xi}{n^{1/2}}\right).
\end{aligned}$$

4. From condition (1.6) it follows immediately that

$$\sum_{k=1}^{\infty} \nu\{T^{-k}(B)\}z^k = \frac{\mu\{B\}z}{1-z} + R(z), \tag{4.1}$$

where $R(z)$ is analytic for $|z| < 1+\epsilon$.

It is now quite easily shown that

$$\lim_{n\to\infty} e^{-i\xi\mu\{B\}n^{1/2}}\beta_n\left(-\frac{i\xi}{n^{1/2}}\right) = e^{-\sigma^2\xi^2/2}, \tag{4.2}$$

where

$$\sigma^2 = \mu\{B\}[1-\mu\{B\}] + 2\mu\{B\}R(1). \tag{4.3}$$

It is harder to prove that

$$\lim_{n\to\infty} n \sup_{1\leq r\leq n}\left|\gamma_r\left(-\frac{i\xi}{n^{1/2}}\right)\right| = 0, \tag{4.4}$$

but once this is done one gets from (3.4) and (4.2) that

$$\lim_{n\to\infty}\frac{\int_B \exp\left(\frac{i\xi}{n^{1/2}}\left[\sum_1^n V(T^k\omega) - n\mu\{B\}\right]\right)d\mu}{\mu\{B\}} = e^{-\sigma^2\xi^2/2} \tag{4.5}$$

and the theorem announced in §1 follows.

It should be pointed out that in proving (4.4) one has to apply (1.6) to all ν_m.

5. The general theorem of this note was motivated by an application to continued fractions.

In this case Ω is the interval $[0, 1]$ and the invariant measure

$$\mu(A) = \frac{1}{\log 2}\int_A \frac{dx}{1+x}\,. \tag{5.1}$$

The transformation Tx in question is given by the formula

$$Tx = \frac{1}{x} - \left[\frac{1}{x}\right] \tag{5.2}$$

and the crucial property (1.6) was proved by Paul Lévy [1].

Our result is thus that *the number of times a specified digit occurs among the first n digits in a continued fraction is asymptotically normally distributed.*

The method used in proving our main result was suggested by the occupation time problem for Markoff chains. The fuller discussion of this connection as well as that of a number of related results will appear elsewhere.

References

1. Paul Lévy, *Theorie de l'addition des variables aleatoires*, Paris, Gauthier-Villars, 1937, page 298 ff.

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