ON THE NONEXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

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Communicated by S. Eilenberg, April 29, 1958

With the usual definitions of homotopy-theory, we have the following theorem.

THEOREM 1. (a) *S^{n-1} is not an H-space unless $n=2$, 4, or 8.*

(b) *There is no element of Hopf invariant one in $\pi_{2n-1}(S^n)$ unless $n=2$, 4, or 8.*

For the context of this question, see [5] (especially pp. 436–438), [4, Chapter VI] and [6, §§20, 21].

This theorem results from reasonings with secondary cohomology operations. It is generally understood that a secondary operation corresponds to a relation between primary operations. One may formalize the notion of a "relation" by introducing pairs (d, z), algebraic in nature, as follows.

Let p be a prime; let A be the Steenrod algebra [2, p. 43] over Z_p. One defines the notion of a graded left module M over the graded algebra A so that $M=\sum_q M_q$ and $A_qM_r \subset M_{q+r}$. For example, let us write $H^q(X)$ for $H^q(X; Z_p)$, $H^*(X)$ for $\sum_q H^q(X; Z_p)$ and $H^+(X)$ for $\sum_{q>0} H^q(X; Z_p)$; then $H^*(X)$ and $H^+(X)$ are graded left modules over A. Let M, N be such modules; one defines the notion of an A-map f: $M \rightarrow N$ of degree r so that $f(M_q) \subset N_{q+r}$.

A pair (d, z), then, is to have the following nature. The first entry d is to be an A-map d: $C_1 \rightarrow C_0$ of degree zero. Here C_0, C_1 are to be modules in the above sense; we require, moreover, that they are locally finitely-generated and free, and that $(C_i)_q=0$ if $q<i$ $(i=0, 1)$. The second entry z is to be a homogeneous element of Ker d.

Let (d, z), then, be a pair of this sort. We call Φ a stable secondary cohomology operation associated with (d, z), if it satisfies the following axioms.

AXIOM (1). $\Phi(\epsilon)$ is defined for each A-map ϵ: $C_0 \rightarrow H^+(X)$ of degree $m \geqq 1$ and such that $\epsilon d=0$.

Such a map ϵ is determined by its values on the elements of an A-base of C_0. It therefore corresponds to a set of elements of $H^+(X)$. In particular, if C_0 is free on one given generator c, we write $u=\epsilon c$; we may thus consider Φ as a function of one variable u, where u runs over a subset of $H^+(X)$. In this case we write $\Phi(u)$ for $\Phi(\epsilon)$.

For the next axiom, set $\deg(z)=n+1$, let f: $C_1 \rightarrow H^+(X)$ run over the A-maps of degree $(m-1)$, and let $Q^{m+n}(d, z; X)$ be the set of elements of the form fz.

AXIOM (2). $\Phi(\epsilon) \in H^{m+n}(X)/Q^{m+n}(d, z; X)$.

For the next axiom, let $g: Y \to X$ be a map.

AXIOM (3). $g^*\Phi(\epsilon) = \Phi(g^*\epsilon)$.

For the next axiom, let (X, Y) be a pair, and let $\epsilon: C_0 \to H^+(X)$ be a map of degree $m \geqq 1$ such that $\epsilon d = 0$ and $i^*\epsilon = 0$. We can now form the following diagram.

$$\begin{tikzcd}
H^+(Y) & H^+(X) \arrow[l, "i^*"'] & H^+(X, Y) \arrow[l, "j^*"'] & H^+(Y) \arrow[l, "\delta"'] & H^+(X) \arrow[l, "i^*"'] \\
 & & C_0 \arrow[ul, "\epsilon"] \arrow[u, "\eta"'] & C_1 \arrow[l, "(-1)^m d"] \arrow[u, "\zeta"'] &
\end{tikzcd}$$

AXIOM (4). $i^*\Phi(\epsilon) = \{\zeta z\} \bmod i^*Q^{m+n}(d, z; X)$.

For the next axiom, let SX be the suspension of X, and let $\sigma: H^+(X) \to H^+(SX)$ be the suspension isomorphism. Let ϵ be as above.

AXIOM (5). $\sigma\Phi(\epsilon) = \Phi(\sigma\epsilon)$.

THEOREM 2. *Given any pair (d, z) (as above), there is at least one stable secondary cohomology operation Φ associated with it (in the sense of the axioms above).*

This theorem is proved by the method of the universal example. The next theorem allows us to study the operations Φ by applying homological algebra (see [**3**]) to the pairs (d, z).

THEOREM 3. (a) *If Φ, Φ' are two operations associated with the same pair (d, z) then there is an element c in $(C_0/dC_1)_n$ such that*

$$\Phi(\epsilon) - \Phi'(\epsilon) = \{\epsilon c\}.$$

(b) *Suppose given d (as above), elements z_t in* Ker d, *and operations Φ_t associated with the pairs (d, z_t). Suppose $z = \sum_t a_t z_t$ $(a_t \in A)$. Then there is an operation Φ associated with (d, z) such that*

$$\sum_t a_t\Phi_t(\epsilon) = \{\Phi(\epsilon)\} \bmod \sum_t a_t Q^{m+n_t}(d, z_t; X).$$

(c) *Suppose given a diagram*

$$\begin{tikzcd}
C_1 \arrow[r, "m_1"] \arrow[d, "d"'] & C_1' \arrow[d, "d'"] \\
C_0 \arrow[r, "m_0"] & C_0'
\end{tikzcd}$$

in which d, d' are as above, and m_0, m_1 are A-maps of degree zero. Let Φ be an operation associated with a pair (d, z). Then there is an operation Φ' associated with $(d', m_1 z)$ such that

$$\Phi(\epsilon' m_0) = \{\Phi'(\epsilon')\}$$

for each $\epsilon' : C_0' \to H^+(X)$ of the sort considered above.

One may show that for operations in one variable, there is a Cartan formula for expanding $\Phi(uv)$, where uv is a cup-product.

We now take $p=2$. We also omit to summarize some work with homological algebra. This work leads us to consider certain pairs (d, z). By applying Theorem 2, we obtain secondary operations $\Phi_{i,j}(u)$ for $0 \leqq i \leqq j, j \neq i+1$. The operation $\Phi_{j,i}(u)$ is of degree 2^i+2^j-1, and it is defined on classes u such that $Sq^{2^r}(u)=0$ for $0 \leqq r \leqq j$.

Let P be complex projective space of infinitely-many dimensions, and let y be a generator of $H^2(P)$ (by which we mean $H^2(P; Z_2)$). We may ask for the values of the operations $\Phi_{i,j}$ in $H^*(P)$. Now, $\Phi_{i,j}(y^r)$ is defined only if $r \equiv 0$ and mod 2^j. Moreover, the degree of $\Phi_{i,j}$ is odd unless $i=0$ and $j>0$; so that $\Phi_{i,j}(y^r)$ lies in a zero group unless $i=0$ and $j>0$. It remains only to consider $\Phi_{0,j}(y^{n2^j})$; this is defined modulo zero.

THEOREM 4.

$$\Phi_{0,j}(y^{n2^j}) = ny^{(n+1/2)2^j} \qquad \text{(mod zero)}.$$

In the proof of this theorem we make essential use of a formula for the composite operation $\Phi_{0,j}Sq^{2^j}$. This formula is proved by applying Theorem 3.

THEOREM 5. *For each $k \geqq 3$ we have a formula*

$$\sum_{i,j; j \leqq k} a_{i,j,k}\Phi_{i,j}(u) = Sq^{2^{k+1}}(u) \qquad \text{(mod } Q).$$

The formula is valid on classes u such that $Sq^{2^r}(u)=0$ for $0 \leqq r \leqq k$, and holds modulo a certain subgroup Q. It is proved as follows. By applying Theorem 3, we obtain a formula

$$\sum_{i,j: j \leqq k} a_{i,j,k}\Phi_{i,j}(u) = \lambda Sq^{2^{k+1}}(u) \qquad \text{(mod } Q)$$

in which $a_{i,j,k} \in A$, and the coefficient λ remains to be determined. Applying the formula to a suitable class u in $H^*(P)$, we determine $\lambda = 1$.

To prove Theorem 1, it is sufficient to prove it for the case $n=2^m$. This case follows immediately from Theorem 5, using the same argument as that used by Adem [1, §4] in the case $n \neq 2^m$.

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