BOOK REVIEWS

Foundations of algebraic topology. By S. Eilenberg and N. Steenrod. Princeton University Press, 1952 (second printing, 1957). 15+328 pp. $7.50.

This book has had a profound influence on the development of topology both before and after its publication. In the five years since its first printing it has become a standard textbook and reference work for anyone interested in topology.

The first course in algebraic topology is usually a difficult one for the student. He faces a mass of unfamiliar algebraic machinery whose motivation is difficult to grasp and whose applicability is appreciated only much later. Realizing this, the authors have adopted an axiomatic approach to the subject of homology theory. Starting with seven easily stated axioms relating algebra and geometry (and assuming only the basic concepts of algebra and point set topology as prerequisites) they show how many important and interesting theorems can be proved directly from these axioms. The axioms themselves are presented without motivation, but their immediate application is intended to make it easier for the student to accept them. Only after the reader has seen the power of the theory is he led into the details of the existence and uniqueness of homology theories.

In order to state the axioms the concept of *admissible category* is introduced. This is a family of pairs (X, A) of topological spaces and continuous maps $f\colon (X, A)\to(Y, B)$ between them which, roughly speaking, contains sufficiently many pairs and maps to state the axioms. Then a *homology theory* on such an admissible category consists of three functions. The first is a function which assigns to every pair (X, A) in the category and every integer q an abelian group $H_q(X, A)$. The second function assigns to every map $f\colon (X, A)\to(Y, B)$ in the category and every integer q a homomorphism $f_*\colon H_q(X, A) \to H_q(Y, B)$. The third function assigns to every pair (X, A) in the category and every integer q a homomorphism $\partial\colon H_q(X, A)\to H_{q-1}(A)$ (where, in the latter group, the pair $(A, 0)$ has been abbreviated to A).

The three functions H_q, f_*, ∂ of a homology theory are required to satisfy seven axioms. The first three assert the functorial (or naturality) properties of f_* and ∂. The others are: the *exactness axiom*, which relates the homology groups of (X, A), X, and A in an exact sequence; the *homotopy axiom*, which asserts that homotopic maps induce the same homomorphism; the *excision axiom*, which asserts that $H_q(X, A)$ depends, to a great extent, only on $X-A$; and the *dimension axiom*, which is a normalization condition requiring that for a single point space P the groups $H_q(P)=0$ for all $q\neq 0$. The *coefficient group* of the homology theory is then defined to be the group $H_0(P)$ where P is a single point space (it follows from the first two axioms that for any two single point spaces P, P' there is an isomorphism $H_0(P)\approx H_0(P')$).

Each of the axioms is a standard theorem of classical homology theory. The fact that these can be used as the starting point for obtaining many of the other results of homology theory is the basic idea underlying the axiomatic approach. Using only the axioms and standard facts of point set topology the authors develop the direct sum theorem and the Mayer-Vietoris sequence, calculate the homology groups of cells and spheres, and prove the Brouwer fixed point theorem, the invariance of domain, and the fundamental theorem of algebra.

Cohomology theory is developed simultaneously with homology theory. Analogous axioms are given for cohomology theory, and, when a theorem of homology theory is derived from the axioms, there is stated, at the end of the section, a similar theorem of cohomology theory whose proof is left to the reader.

The first chapter presents the axioms and some of their immediate consequences. The second and third chapters develop the homology theory of simplicial complexes from the axioms and prove the basic *uniqueness theorem* that any two homology theories with isomorphic coefficient groups are isomorphic on the category of triangulable pairs. Chapters IV, V, and VI are concerned with categories and functors, chain complexes, and formal homology theory of simplicial complexes, leading to the *existence theorem* that homology theories with arbitrary coefficient groups exist on the admissible category of triangulable pairs. Chapter VII develops singular homology theory, thus proving the existence of homology theories on the category of all pairs (X, A) and maps of such pairs.

Chapters VIII, IX, and X are concerned with Čech homology theory and its special features. This leads to another proof of the existence of homology theories. It is also shown that the Čech groups satisfy the *continuity axiom*, which states that for compact pairs (X_α, A_α) forming an inverse system then $H_q(\liminf (X_\alpha, A_\alpha)) \approx \liminf H_q(X_\alpha, A_\alpha)$. A uniqueness theorem is proved for continuous homology theories on compact pairs. Chapter XI gives some applications to euclidean spaces. Most of these applications follow directly from the axioms, but a section on manifolds uses, in addition, the continuity axiom.

Whether one agrees that the axiomatic approach is a good one for beginning students or not, there is much to recommend the book for use as a text (most suitable, perhaps, for a second year graduate course). The treatments of the singular and Čech theories are modern, complete, and quite readable by themselves. Diagrams of homomorphisms, which are used so frequently today, were first systematically used in this book, both to motivate proofs and to assist the reader in following arguments. Each chapter of the book begins with an introduction stating what the chapter covers and how the material fits into the general scheme of the book. Notes are at the end of the chapter. These discuss the historical development of the subject and its relations to other topics. References to the literature are also found in these notes. Each chapter is followed by a set of exercises. Some of these are easy and some more difficult but most of them are interesting, and the student who works his way through them will learn a great deal.

Since its publication the terminology and notation of the book has been almost universally adopted by topologists. The axioms have led to cleaner proofs of many theorems and increased their generality at the same time. In addition, the axioms have been applied to prove new results. One of the most recent of these applications is the theorem proved by Dold and Thom (C. R. Acad. Sci. Paris vol. 242 (1956) pp. 1680–1682) to the effect that the qth homotopy group of the infinite symmetric product of a polyhedron X is isomorphic to the qth homology group of X. They prove this by showing that the homotopy groups of the infinite symmetric product of X, regarded as functions of X, satisfy the axioms, whence the result follows from the uniqueness theorem for polyhedra.

The book contains no discussion of cup products or cross products. This was to be included in a projected second volume, which was to contain also a treatment of cell complexes and the practical calculation of the homology groups of such spaces. It is to be hoped that the authors have not abandoned their plan to write this second volume. Such a continuation of the present useful book would be a welcome and worthwhile contribution to the mathematical literature.

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Vorlesungen über Himmelsmechanik. By Carl Ludwig Siegel. Springer, Gottingen, 1956. 9+212 pp. DM 29.80. Bound DM 33.

The appearance of this remarkable book is certainly one of the great mathematical events of the century. Written on the subject matter which is the mother field of modern mathematics and spar-