A CLASS OF LATTICE ORDERED ALGEBRAS[1]

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1. Our purpose is to characterize those lattice ordered algebras which may be represented as algebras of Carathéodory functions. This work is, accordingly, a sequel to [1] where the same problem was considered for lattice ordered groups. The rings considered here are more restrictive than those of Birkhoff and Pierce in [2], where an "F-ring" is shown to be isomorphic to a subring of the direct union of totally ordered rings (but the multiplication in [2] is not necessarily that which may be expected for functions; indeed, all products may be zero. In our case, the axioms compel the algebra multiplication to conform to that of the Carathéodory functions). Brainerd [3] has considered a class of algebras which have function space representations, but his emphasis is different from ours.

2. In this section, we define a Carathéodory algebra. Let B be a relatively complemented distributive lattice. Let E be the set of forms $f=a_1\alpha_1+\cdots+a_n\alpha_n$, where $\alpha_i\in B$, a_i real, $i=1,\cdots,n$. With $f\geqq 0$ if $a_i\geqq 0$ for all i, and addition and multiplication defined by $f+g=\sum_{i=1}^n\sum_{j=1}^m(a_i+b_j)(\alpha_i\cap\beta_j)+\sum_{i=1}^n a_i(\alpha_i-\bigcup_{j=1}^m\beta_j)+\sum_{j=1}^m b_j(\beta_j-\bigcup_{i=1}^n\alpha_i)$ and $fg=\sum_{i=1}^n\sum_{j=1}^m a_ib_j(\alpha_i\cap\beta_j)$ where $f=\sum_{i=1}^n a_i\alpha_i$ and $g=\sum_{j=1}^m b_j\beta_j$, E is a lattice ordered algebra, which we call the algebra of elementary Carathéodory functions. Let $\overline{E}$ be the conditional completion of E. $\overline{E}$ is the set of bounded Carathéodory functions. In order to define the general Carathéodory function, we need the notion of carrier. In a lattice ordered group, for every $x\geqq 0$, $y\geqq 0$, we say $x\sim y$ if $x\cap z=0$ when and only when $y\cap z=0$. The equivalence classes obtained in this way are called carriers (filets by Jaffard [4]) and form a relatively complemented distributive lattice. The equivalence class to which x belongs is called the carrier of x. In $\overline{E}$, consider pairwise disjoint sequences $\{f_n\}$ whose carriers have an upper bound, and consider the formal sums $\sum f_n$. With order, addition, and multiplication defined appropriately, these formal sums constitute a lattice ordered algebra—the Carathéodory algebra C generated by B. (For details on related matters see [5; 6] and [1].)

3. Let R be an archimedean lattice ordered algebra. Then R is a lattice with positive cone P such that x, $y\in P$, $a\geqq 0$ real, implies

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$x+y$, xy, $ax\in P$, and if $x, y\in P$, $y>0$, implies there is a real $a\geqq 0$ with $x-ay\notin P$. We say that R is totally complete if

(a) R is conditionally complete.

(b) every sequence of pair-wise disjoint elements in P, whose sequence of carriers has an upper bound, itself has an upper bound; hence, a least upper bound.

In addition to the archimedean hypothesis, the following condition is important for us.

A. If x, y, z are in P(i.e., $x\geqq 0$, $y\geqq 0$, $z\geqq 0$) then $(xy)\cap z=0$ if and only if $x\cap y\cap z=0$.

It is not hard to see that the Carathéodory algebra C is totally complete and satisfies A.

4. Before considering the main problem, we point out that for every totally complete vector lattice R, multiplication may be defined so that R is an algebra satisfying A. We outline the procedure.

Let $[u_\alpha]$ be a generalized weak unit [1] in R. Then, for every carrier α, there is a unique u_α with carrier α, and for every α, β we have $u_\alpha\cap u_\beta=u_{\alpha\cap\beta}$ and $u_\alpha\cup u_\beta=u_{\alpha\cup\beta}$. For every $x>0$ there is, by the total completeness of R, a pairwise disjoint sequence $\{u_{\alpha_n}\}$ and a sequence $\{a_n\}$ of positive reals, such that $\sup a_n u_{\alpha_n}\geqq x$. For every $x>0$, $y>0$ let u_{α_n}, a_n be as above relative to x and v_{β_n}, b_n as above relative to y. Let $\xi=\sup (a_n u_{\alpha_n})(b_m v_{\beta_m})$. Then define $xy=\inf \xi$ for all ξ obtained in this way. For any $x, y\in R$, define $xy=x^+y^++x^-y^--x^+y^--x^-y^+$. It can then be shown that R is an algebra satisfying A. Moreover, if R has a weak unit, the resulting algebra has an identity.

5. We now let R be a totally complete lattice ordered algebra, satisfying A.

LEMMA 1. *If $x\geqq 0$, $y\geqq 0$ then $xy=0$ if and only if $x\cap y=0$.*

LEMMA 2. *If $x\geqq 0$ then x and x^2 have the same carrier.*

PROOF. $x\cap y=0$ *implies* $x\cap x\cap y=0$ implies $x^2\cap y=0$. Conversely, $x^2\cap y=0$ implies $x\cap x\cap y=0$ implies $x\cap y=0$. More generally,

LEMMA 2′. *If x, $y\geqq 0$ have the same carrier, then xy also has this carrier.*

COROLLARY 1. *Every carrier is a semi-ring.*

Since R is conditionally complete, for every $x, y\in R$, the projection y_x of x on y is defined.

LEMMA 3. $xy=xy_x$.

The next lemma is important for us.

LEMMA 4. *If* $x>0$ *there is* $y>0$ *with* $yx \geqq x$ *and* $z>0$ *with* $zx \leqq x$.

We outline the proof. From Lemma 2, the supremum of the carriers α_n of $w_n=(nx^2-x)^+$ is the carrier of x. Let $\beta_n=\alpha_n-\alpha_{n-1}$ and let z_n have carrier β_n. If $y_n=(nx)_{z_n}$, the y_n are pair-wise disjoint. By the total completeness of R, sup $y_n=y$ exists. Then $yx \geqq x$. The proof of the second part is similar.

DEFINITION. For every $x \geqq 0$, $u(x)=\inf\ [y \mid yx \geqq x]$ and $\bar{u}(x)=\sup\ [y \mid yx \leqq x]$.

LEMMA 5. *For every* $x \geqq 0$, $x=u(x)x=\bar{u}(x)x$.

PROOF. $u(x)x \geqq x$. If $u(x)x>x$ there is $z>0$ with $zx<u(x)x-x$, whereby $(u(x)-z)x>x$, which is impossible.

LEMMA 6. $[u(x)]^2=u(x)$ *and* $[\bar{u}(x)]^2=\bar{u}(x)$.

PROOF. $[u(x)]^2x=u(x)[u(x)x]=u(x)x=x$ so that $[u(x)]^2 \geqq u(x)$. Similarly, $[\bar{u}(x)]^2 \leqq \bar{u}(x)$. But $\bar{u}(x)x=x$ implies $\bar{u}(x) \geqq u(x)$. However, $\bar{u}(x) \leqq u(x)$.

COROLLARY 2. $u(x)=\bar{u}(x)$.

LEMMA 7. *The carriers of* x *and* $u(x)$ *are the same.*

PROOF. By condition A.

LEMMA 8. *If* x *and* y *have the same carrier then* $u(x)=u(y)$.

PROOF. If $0<x<z<y$ and $x^2=x$, $y^2=y$ then $z^2=z$. Let α be the carrier of x and y. If $u(x) \neq u(y)$, there is $\beta<\alpha$ and $k<1$ such that, say, $k(u(x))_w<(u(y))_w$, where w has β as carrier. But then $[k(u(x))_w]^2=k(u(x))_w$ and $k(u(x))_w=(u(x))_w$. This is impossible.

Thus there is a one-one correspondence $\alpha \rightarrow u_\alpha$ between the carriers and idempotents. There is a unique left identity for every carrier relative to the carrier; there is also a unique right identity.

LEMMA 9. *For every* α, *the associated right and left identities are equal.*

PROOF. Both are idempotents. The proof is then as for Lemma 8. We summarize:

THEOREM 1. *A totally complete lattice ordered algebra* R *satisfying* A *has a unique idempotent* u_α *with carrier* α, *for every* α. *The idempotent* u_α *is an identity (left and right) for all* $x \in R$ *whose carrier is* $\leqq \alpha$.

COROLLARY 3. *The family* $[u_\alpha]$ *is a generalized weak unit in* R.

Proceeding as in [1], the algebra R can be reconstructed from the u_α and a one-one correspondence obtained between the elements of R and those of the space C of Carathéodory functions generated by the relatively complemented distributive lattice B of carriers in R. In this correspondence, each element $a_1u_{\alpha_1}+ \cdots +a_nu_{\alpha_n}\in R$ is mated with the element $a_1\alpha_1+ \cdots +a_n\alpha_n\in C$. It is then a routine matter to check that this correspondence preserves order, addition, and multiplication. We thus have:

THEOREM 2. *A lattice ordered algebra is isomorphic with the algebra C of Carathéodory functions generated by a relatively complemented distributive lattice if and only if it is totally complete and satisfies A; i.e., for x, y, $z\geqq 0$, $(xy)\cap z=0$ if and only if $x\cap y\cap z=0$.*

The following conditions are closely related to A.

A′. If x, $y\geqq 0$, then $xy=0$ if and only if $x\cap y=0$.

A″. R is an F-ring with no nonzero nilpotents.

Indeed, M. Henriksen has shown (oral communication) that conditions A, A', A'' are equivalent. Using this fact, and a completion theorem of Nakano [7] we obtain:

COROLLARY 4. *An archimedean lattice ordered algebra which satisfies A, and is such that* $\inf S=0$ *and* $x\geqq 0$ *implies* $\inf xS=0$, *is isomorphic with a subalgebra of a Carathéodory algebra.*

We also obtain the following fact, which was proved in a different way for F-rings by Birkhoff and Pierce.

COROLLARY 5. *An archimedean lattice ordered algebra which satisfies A has commutative multiplication.*

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