denotes the nth prime, then $p_{n+1}<2p_n$. Legendre conjectured, but no one has ever proved, that $p_{n+1}-p_n<p_n^{1/2}$ for all sufficiently large n. Hoheisel in 1930 established the existence of a number a, $1-1/33000<a<1$, such that $p_{n+1}-p_n<p_n^a$. The exponent a was successively diminished by Heilbronn in 1933, by Tschudakoff in 1936 and by Ingham in 1937. Ingham obtained $a=5/8$ and also a somewhat smaller value. The proofs of these and related results make use of the theory of the density of the zeros of the zeta function. In Chapter Nine the machinery of this theory is developed. Applications are given also to the work of Linnik (1943, 1945), Rodosskii (1949), Tatuzawa (1950) and Haselgrove (1951) on the distribution of primes in "short" arithmetic progressions. Further applications concern the estimation of $\zeta(1/2+it, w)$.

The crowning achievement of the last chapter is the deep theorem of Linnik: Let $k\geqq 2$, $(l, k)=1$, $l<k$, and let $p_1(k, l)$ be the smallest prime in the arithmetic progression $nk+l$, $n=1, 2, \cdots$. Then there exists a constant C independent of k such that $p_1(k, l)<k^C$. The awe-inspiring proof involves forty pages and twenty-one lemmas.

The book closes with an Appendix. This contains a brief summary of pertinent theorems and formulae from the theory of functions.

The author is to be congratulated for having written an important and valuable book. The House of Springer is to be congratulated on a superb example of the art of mathematical printing.

ALBERT LEON WHITEMAN

Neue topologische Methoden in der algebraischen Geometrie. By F. Hirzebruch. Ergebnisse der Mathematik und ihrer Grenzgebiete, New Series, vol. 9. Springer, 1956. 165 pp. DM 30.80.

This book, devoted to the topological transcendental theory of algebraic varieties over the complex field, should rank with Lefschetz's *L'Analysis situs et la géométrie algébrique*, Paris, 1924, and Hodge's *Harmonic integrals*, Cambridge, 1941, as a milestone in the development of the theory. While topology plays the essential rôle in Lefschetz's book and Hodge's main tool is harmonic differential forms, this book is characterized by the diversity of deep and difficult results which the author drew for his use. These include, among others, Todd's genus, Thom's algebra, and Kodaira's work on complex manifolds. Sheaves (or stacks or faisceau in French and Garbe in German) and analytic bundles with their characteristic classes are the pillars on which the main result is built.

The main result, which is not proved until the very end of the book, is the Riemann-Roch Theorem for nonsingular complex algebraic varieties. The classical Riemann-Roch Theorem is concerned with the following problem: On a compact Riemann surface there is associated to each meromorphic function f a divisor $\Delta(f) = \sum m_i P_i$, which is a formal sum of points P_i with integer coefficients m_i equal to the orders of f at P_i; m_i is positive if P_i is a zero and negative if P_i is a pole. In general, a divisor $D = \sum n_j Q_j$ is a sum of points Q_j in which only a finite number of the cofficients n_j are nonzero. The sum $\deg(D) = \sum n_j$ is called the degree of the divisor. A divisor is said to be $\geqq 0$ if all $n_j \geqq 0$. Divisors can be added and subtracted in an obvious way. Given a divisor $D = \sum n_j Q_j$, all meromorphic functions f on the Riemann surface such that $\Delta(f) + D \geqq 0$ form a complex vector space. (The condition means of course that f has a zero of order $\geqq -n_j$ at a point Q_j with $n_j < 0$ and a pole of order $\leqq +n_j$ at a point Q_j with $n_j > 0$ and is regular at all other points.) The classical Riemann-Roch Theorem says that the dimension of this vector space is

$$1 + \dim |D| = \deg(D) - g + i + 1, \tag{1}$$

where g is the genus of the Riemann surface and $i-1$ is equal to the dimension of the divisor $K-D$, K being the canonical divisor. If D is ample, e.g., if D consists of a sufficient number of points with nonzero coefficients, then $i=0$ and formula (1) gives precise information on $\dim |D|$.

The author took (correctly) the view that this is really a theorem identifying two radically different numbers. Let M be an algebraic variety of dimension n and W an analytic vector bundle over M, having as structural group the complex general linear group $GL(q, C)$ in q variables. W will be called a line bundle if $q=1$. Denote by $\Omega(W)$ the sheaf of germs of holomorphic cross-sections of W and by $H^i(M, \Omega(W))$, $0 \leqq i \leqq n$, the ith cohomology group. The most important of these cohomology groups is $H^0(M, \Omega(W))$, which is by definition the vector space of all global holomorphic cross-sections of the bundle W and which therefore contains some of the most valuable information on W. It is, however, the alternating sum

$$\chi(M, W) = \sum_{0 \leq i \leq n} (-1)^i \dim H^i(M, \Omega(W)), \tag{2}$$

which has notable properties. For a constant coefficient sheaf the sum (2) gives the classical Euler-Poincaré characteristic.

The second number is related to the analytic structure in an entirely different way. In fact, let c_i, $1 \leqq i \leqq n$, be the Chern classes of the tangent bundle of M, and d_j, $1 \leqq j \leqq q$, be the Chern classes of W. Introduce formally the quantities γ_i, $1 \leqq i \leqq n$, and δ_j, $1 \leqq j \leqq q$, by the relations

$$(3)\qquad \begin{aligned} 1 + \sum c_i &= \prod (1 + \gamma_i), \\ 1 + \sum d_j &= \prod (1 + \delta_j). \end{aligned}$$

Define

$$(4)\qquad T(M, W) = \kappa_{2n}\left[(e^{\delta_1} + \cdots + e^{\delta_q}) \prod_{i=1}^{n} \frac{-\gamma_i}{\exp(-\gamma_i) - 1}\right],$$

where the expression inside the brackets is symmetric in γ_i and δ_j and hence can be expressed as a power series in c_i and d_j with rational coefficients, while the symbol κ_{2n} means the value of this cohomology class over the fundamental homology class of M. The Riemann-Roch-Hirzebruch Theorem says that

$$(5)\qquad \chi(M, W) = T(M, W).$$

In particular, this implies that $T(M, W)$ is an integer, which is by no means clear from the definition. If the bundle W is analytically a Cartesian product, both $\chi(M, W)$ and $T(M, W)$ depend only on M and we will denote them by $\chi(M)$ and $T(M)$ respectively. $\chi(M)$ is essentially the arithmetic genus of M and $T(M)$ is the Todd genus. By some analytic manipulation one can see that (5) will follow from the particular case

$$(6)\qquad \chi(M) = T(M).$$

But the proof of (6) is not less difficult than that of (5).

We mention in passing that (5) contains (1) as a particular case. In fact, a divisor on a Riemann surface defines a line bundle whose Chern class d_1 has the property that $\kappa_2(d_1)$ is equal to the degree d of the divisor. Formula (5) gives

$$(7)\qquad \dim H^0(M, \Omega(D)) - \dim H^1(M, \Omega(D)) = d - g + 1,$$

which is equivalent to (1) by Serre's duality theorem.

Returning to the proof of (6) it is natural to observe that the advantages are obvious if γ_i are actually cohomology classes. This is the case when M is a split manifold, which means that the tangent bundle is analytically equivalent to a bundle with the triangular group as structural group, or, in other words, there is an analytic field of flags (i.e., a sequence of linear subspaces $L_1 \subset L_2 \subset \cdots \subset L_n$ in the tangent space) over M. An important example of a split manifold is the bundle B of all flags over M. If we take the polynomial

$$(8)\qquad C(t) = 1 + \sum c_i t^i$$

with the Chern classes of M as coefficients, then this bundle B has the interesting property, reminiscent to the extension of an algebraic number field, that the inverse image of (8) in B decomposes into a product of linear factors. The author proves that $\chi(B)=\chi(M)$, $T(B)=T(M)$, so that it suffices to prove (6) for a split manifold.

The author's idea is to generalize $\chi(M, W)$, $T(M, W)$ to involve a a parameter. In the simplest case when the bundle W is not involved, they are defined as follows: Let Ω^p be the sheaf of germs of holomorphic p-forms over M. Then we define

$$\chi_y(M) = \sum_{0\leq i,p\leq n} (-1)^i \dim H^i(M, \Omega^p)y^p. \tag{9}$$

On the other hand, let

$$Q(y; z) = \frac{z(y+1)}{\exp(z(y+1)) - 1} + z \tag{10}$$

and

$$T_y(M) = \kappa_{2n}\left[\prod_{1\leq i\leq n} Q(y; \gamma_i)\right]. \tag{11}$$

The introduction of the parameter y combines several known invariants. In fact, by definition, $\chi_0(M)=\chi(M)$. The Hodge theory of harmonic differential forms shows that $\chi_{-1}(M)$ is the ordinary Euler-Poincaré characteristic of M and that $\chi_1(M)$ is the index of M. The latter is defined to be zero if n is odd and to be equal to the number of positive eigenvalues minus the number of negative eigenvalues of the intersection matrix of (real) dimension n of M, if n is even. From the Gauss-Bonnet formula for compact Kähler manifolds one can prove $\chi_{-1}(M)=T_{-1}(M)$. But in order to prove $\chi_y(M)=T_y(M)$ for all y, it is necessary and sufficient to prove their equality for a value of $y\neq -1$. The author's index theorem says that $\chi_1(M)=T_1(M)$, which therefore forms the bridge between the χ_y-theory and the T_y-theory.

The index theorem, which can be described as equating the index of a compact oriented manifold to a certain Pontryagin number, is a consequence of Thom's algebra. Its origin can be traced to a result of Pontryagin, which says that the index of a bounding manifold (i.e., one which is the boundary of a manifold of one dimension higher) is zero. Thom realized that in studying the characteristic numbers of compact manifolds it is important to introduce an equivalence relation, by calling two manifolds equivalent, when their difference is a bounding manifold. Defining sum as union and product as the Cartesian product, such equivalence classes of manifolds are made into an algebra. Thom proved that the tensor product of this algebra with the field of rational numbers has as a multiplicative base the complex projective spaces of different dimensions. The index theorem follows from the Thom theory by the simple observation that it is true for the complex projective spaces.

The above is just a description of some of the important ideas which enter into the proof, with no attempt to give an outline. The use of the index theorem is effective, but probably unnatural. Even if a more direct proof of the Riemann-Roch Theorem is found, many of the ideas in this book should be found useful in other problems. The field certainly deserves further study. In fact, the two natural directions of extension are either algebraic varieties over finite fields and with singularities allowed or general complex manifolds.

Because of the presence of the high-dimensional cohomology groups in (5) it may be felt that the theorem is probably less effective in applications than the classical case. For algebraic surfaces ($n=2$) the theorem implies the classical Riemann-Roch inequality and is sufficient for most purposes. (An exception is the theorem that if the plurigenera P_4 and P_6 are zero, then the surface is birationally equivalent to a ruled surface. For the proof of this theorem, the mere knowledge of the inequality is not sufficient; it is necessary to have some information about the superabundance of certain linear systems $|D|$, i.e., precisely about the value of dim $H^1(M, \Omega(D))$.)

The book uses many of the deep results in different branches of mathematics, and may cause difficulty even to readers with a good background. One should realize, however, that this is essentially an original paper. For such the introductory material is ample; it is also well written. If the reader succeeds in reaching the summit, the panorama is highly recommendable.

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