RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

PRODUCTS OF SYMMETRIES

BY PAUL R. HALMOS AND SHIZUO KAKUTANI

Communicated January 25, 1958

A (bounded) operator Q on a (complex) Hilbert space H is a *symmetry* if it is a unitary involution, i.e., if $Q^*Q=QQ^*=1$ (=the identity operator on H) and $Q^2=1$. In connection with his studies of the infinite-dimensional analogues of the classical groups, R. V. Kadison has asked us which operators can be represented as (finite) products of symmetries. The purpose of this note is to give a precise answer to Kadison's question.

THEOREM 1. *If H is infinite-dimensional, then every unitary operator on H is the product of four symmetries.*

PROOF. We need the auxiliary result that if U is a unitary operator on an infinite-dimensional Hilbert space H, then there exists a (closed) subspace H_0 of H such that H_0 reduces U and such that $\dim H_0 = \dim H_0^{\perp}$. This result holds, in fact, for an arbitrary normal operator on H. Since the proof is a straightforward application of the spectral theorem, and since the proof for a typical special case (namely, for Hermitian operators) has already appeared in the literature,[1] we do not present it here.

We apply the auxiliary result to the (unitary) operator on $H_0^{\perp}$ obtained by restricting U to $H_0^{\perp}$ and obtain thus a subspace H_1 (of $H_0^{\perp}$) such that H_1 reduces U and such that $\dim H_1 = \dim (H_0^{\perp} \cap H_1^{\perp})$. Proceeding inductively, we obtain an infinite sequence $\{H_n\}$ of orthogonal subspaces (of H) such that each H_n reduces U and such that every H_n has the same dimension. If the intersection of the orthogonal complements of all the H_n is not trivial, it can be amalgamated to H_0; it follows that H is the direct sum of countably many equidimensional subspaces each of which reduces H. By suitably renumbering the terms of this sequence, we may assume that the index n runs through all (not necessarily non-negative) integers.

Relative to the fixed direct sum decomposition $H=\sum_n H_n$, we

[1] Paul R. Halmos, *Commutators of operators*, Amer. J. Math. vol. 74 (1952) pp. 237–240; see Lemma 3 on p. 239.

define a *right shift* as a unitary operator S such that $SH_n = H_{n+1}$ for all n, and we define a *left shift* as a unitary operator T such that $TH_n = H_{n-1}$ for all n. The equi-dimensionality of all the H_n guarantees the existence of shifts. If S is an arbitrary right shift, we write $T = S^*U$. Since $TH_n = S^*UH_n = S^*H_n = H_{n-1}$ for all n, it follows that T is a left shift. Since $U = ST$, we have proved that every unitary operator on H is a product of two shifts; we shall complete the proof of the theorem by showing that every shift is the product of two symmetries.

Since the inverse (equivalently, the adjoint) of a left shift is a right shift, it is sufficient to consider right shifts. Suppose then that S is a right shift; let P be the operator that is equal to S^{1-2n} on H_n and let Q be the operator that is equal to S^{-2n} on H_n for all n. If $x \in H_n$, then $Qx = S^{-2n}x \in S^{-2n}H_n = H_{-n}$, so that $PQx = PS^{-2n}x = S^{1-2(-n)}S^{-2n}x = Sx$. The proof of Theorem 1 is complete.

To what extent is Theorem 1 the best possible result along these lines? The hypothesis of infinite-dimensionality clearly cannot be omitted. Indeed, if H is finite-dimensional, then the concept of determinant makes sense. Since the determinant of a symmetry is ± 1, it follows that no (unitary) operator with a nonreal determinant can be the product of symmetries. Equally clearly, the conclusion of the theorem cannot be strengthened so as to apply to nonunitary operators, because a product of unitary operators (and, in particular, of symmetries) must be unitary. The only conceivable improvement, therefore, is quantitative: possibly every unitary operator is a product of three symmetries. We conclude by showing that this is not so.

THEOREM 2. *On every Hilbert space H there exists a unitary operator U that is not the product of three symmetries.*

PROOF. Let c be a complex cube root of unity and let U be $c \cdot 1$. The operator U belongs to the center of the group of all unitary operators on H; the order of U in that group is exactly three. The remainder of our proof has nothing to do with operator theory; we shall show that, in every group G, a central element of order three is not the product of three elements of order two. More precisely, we show that if G is a group, if u is a central element in G, and if $u = xyz$ with $x^2 = y^2 = z^2 = 1$, then $u^4 = 1$. The proof consists of a simple computation:

$$\begin{aligned} u^4 &= uxuyuz = u(xu)\cdot y(uz) = u(yz)\cdot y(xy) \\ &= y(uz)\cdot y(xy) = yxy\cdot yxy = 1. \end{aligned}$$

UNIVERSITY OF CHICAGO AND
YALE UNIVERSITY