Fundamental concepts of algebra. By Claude Chevalley. New York, Academic Press, 1956. 8+241 pp. $6.80.

Chevalley has written a text-book, and his mathematical personality permeates every paragraph. Readers of his *Algebraic functions of one variable* who agreed at the time with André Weil's dictum "algebraic austerity can go no further" may decide that a counterexample has been produced. The book is tight, unified, direct, severe; relentlessly and uncompromisingly it pursues its ends: out of the simplest basic notions of algebra to build up with perfect precision the theory of the multilinear algebras of modules and to discuss those particular multilinear algebras which have found applications in topology and differential geometry. Group and ring theory are down to the irreducible minimum, field theory is completely absent; in their place, modules and their tensor products and the algebras one constructs from them: tensor, exterior, symmetric. The unity is monolithic. Gone is the discursive rambling of previous texts. This one marches unswerving and to its own music. It is presented by Chevalley as a serious effort to "adapt modern algebra teaching to present-day requirements"; since it represents thereby the first real departure in English from the van der Waerden tradition in first-year graduate algebra texts, it should be considered in some detail.

The general approach to the subject matter is that of Bourbaki's first three algebra chapters, but there are significant differences in content and treatment (Chevalley is often more general). As for the style, Bourbaki emerges from the comparison a warm, compassionate, and somewhat elderly gentleman.

The first chapter of the book is an Overture: in these first twenty pages are set forth the major themes of the entire book. They are devoted to the monoid (*née* semigroup with identity) and her cortege —submonoid, quotient monoid and homomorphism, product and free monoid. Two basic modern techniques appear and are emphasized from the very beginning here. One is the proof that A and B are isomorphic by constructing opposing maps f and g such that $f \circ g$ and $g \circ f$ are the identity. The other is the universal mapping property characterization, here used to define the free monoid and soon to be ubiquitous. This emphasis on mappings is one of the most characteristic features of the new algebra; the older "identifications" are now explicit natural mappings, and what earlier required brow-wrinkling now needs only diagram-chasing. So Chevalley appropriately puts in here two little diagrams and three little paragraphs explaining them. In view of the importance of diagrams to this sort of algebra, probably a few more should have been included later on as a suggestion to the novice how he might best follow some of the more complicated theorems.

Chapter two reviews the same concepts for a group, adding a discussion of homogeneous spaces and stability sets, a basic concept curiously ignored by algebra texts. The central chapter three defines a ring (with identity) and passes at once to the notion of a module M, an additive group with a ring R of operators. Insofar as possible M is general (not finitely generated or free) and R is not assumed commutative. Semi-simple modules are those in which every submodule is a direct summand; these are decomposed as usual. Hom (M, N) and $M \otimes N$ are introduced, then R is made commutative so multilinear maps can be discussed. The pages at this point start to look pretty gruesome, though the worst-looking lemmas turn out to be sheep in wolves' clothing—things like the general distributivity law for multilinear maps. The author's insistence on perfect precision of utterance make some rather intricate verbal pirouettes necessary to finally produce, say, the multinomial formula. Modules fly thick and fast, and one begins to understand what a poor Strasbourg goose whose liver is being fattened up for pâté must feel like. But after one more section of technique discussing the modules naturally derived from M by changing the ground ring R to a homomorphic image R', we can finally relax with vector spaces, matrices and their equivalence transformations, and linear maps. Here the insistence on unity both of methods and ideas means that certain classical topics —the basis theorem for a module when R is a principal ideal ring and the resulting canonical forms for a matrix, for example—cannot be included. The diagonalizable matrices are however accessible and are discussed; determinants, of course, must wait for the exterior algebra.

After a sidelong glance at the general (non-associative) algebra, we pass into the final and lengthiest section of the book: a treatment of four special associative algebras is given which provides a thorough working out of the basic techniques the author has been insisting on from the beginning. The pattern is somewhat the same for three of them, the tensor, exterior, and symmetric algebras: characterization by a universal mapping property, existence, and basic properties with respect to the module operations of chapter three. Everything of course is done as invariantly as possible, and extraordinary measures are occasionally adopted to avoid completely any ad hoc computation, no matter how simple.

The handling of the exterior and Grassmann algebras is interesting. Chevalley makes a distinction between them because the underlying module M is general. The exterior algebra $E(M)$ is the usual one universal with respect to maps ϕ such that $[\phi(m)]^2=0$, while the Grassmann algebra is defined to be the dual module $[E(M)]^*$, endowed with a multiplication defined by

$$\phi \wedge \psi = (\phi \otimes \psi) \circ U$$

where U is the map of $E \to E \otimes E$ induced by the diagonal map of $M \to M \times M$. There is a natural homomorphism of $E(M^*)$ into $[E(M)]^*$ which is an isomorphism if M is free with a finite basis so that the two algebras are isomorphic if M is in particular a finite dimensional vector space. This is the classical case, and Chevalley discusses it subsequently in some detail.

Another original feature of this last chapter is the treatment of the Pfaffian of a skew-symmetric bilinear form. Classically one takes a skew-symmetric matrix A and shows in a few lines by direct verification that its determinant is the square of another polynomial in the matrix entries, called the Pfaffian of A, $Pf(A)$. A longer and less explicit, but also less mysterious procedure is to use first an analogue of the Gram-Schmidt process to select a basis of the vector space (or free module), relative to which the matrix has a simple canonical form whose determinant is obviously a square. Then one takes the original skew-symmetric matrix to have independent transcendental entries, and the theorem follows easily [see, for example, Artin, *Geometric algebra*]. Chevalley proceeds as follows. The original skew-symmetric bilinear form γ on M gives as usual a map of $M \to M^*$; let $\gamma_x \in M^*$ be the image of $x \in M$. He shows that the linear form γ_x extends uniquely to a derivation d_x of degree -1 of the exterior algebra E over M. If we now let L_x denote left multiplication by x and define

$$\Omega(x_1 x_2 \cdots x_p) = (L_{x_1} + d_{x_1}) \circ \cdots \circ (L_{x_p} + d_{x_p}) \ [1],$$

then Ω extends linearly to be an automorphism of E. Now for $t \in E$, let $\epsilon(t)$ be the "constant" term of t, and let Γ be the natural extension of the bilinear form γ to $E \times E$. Then Chevalley proves that

$$\Gamma(t, t) = (\epsilon(\Omega(t)))^2,$$

so that if in particular M is a finite dimensional vector space, and t is the top-dimensional "volume element" of E, $e_1 e_2 \cdots e_n$, then the above relation is exactly

$$\det A = (Pf(A))^2.$$

(The explicit polynomial expression for the Pfaffian is tucked away as a line in one of the exercises, and the above used as the definition.)

The chapter finishes with a discussion of the symmetric algebra on M. If in particular M is free, the symmetric algebra is just a polynomial ring, and this gives Chevalley another chance to use derivations; this time they lead somewhat less excitingly to Taylor's formula in n variables, with which the book concludes.

There are a number of exercises after each chapter—well over a hundred in all. Almost none of then are routine, nor are there many in the nature of specific examples; most of them are not easy. Many provide significant extensions and applications of the theory, and are the equivalent of many additional pages of text. Such, for instance, would be the exercises on the derived groups of a group, projective and injective modules, projective limits, quadratic forms and Clifford algebras, and representations of Lie algebras.

The importance of multilinear algebra as it appears here is hardly to be questioned, lying as it does at the bottom of the modern bundle and cohomological methods in algebra, topology, and differential geometry, and this book is the only English exposition I know of. Bourbaki is easier to read, but there is a forbidding amount of it, not all relevant. Chevalley is more compact and would be a fine book to precede say the Cartan-Eilenberg *Homological algebra* or many of the Cartan and Sophus Lie seminar notes from Paris. Except for some of the material in the last chapter, all of it is genuinely fundamental: the prospective reader can at least be assured that his sweat will not be wasted. This basic and essential *usefulness* of the book should be kept in mind as overshadowing any critical remarks made below.

In considering the treatment the book presents of its subject, one must recognize that it is extremely abstract, and the level of abstraction each man likes in his mathematics is as personal a taste as the amount of perfume he likes on his wife. When linear transformations made their comeback over matrices, it was easy to point to the shortened proofs and to the gain in clarity resulting from the triumph of abstraction over algorithm. An intense sans-culottism has since made the subject perhaps a bit top-heavy in concepts (after all, we still do have a multiplication table). Each reader will have to decide for instance whether Chevalley's seven page intrinsic treatment of the Pfaffian here is the height of beauty and elegance, or of absurdity, and if the former, whether the associated aroma is that of ripe bananas or of freshly-roasted coffee.

Considering the book from the standpoint of the student who wishes to learn multilinear algebra, an outstanding feature is the patient and insistent way in which the basic techniques are applied over and over again. One cannot emphasize too strongly the beautiful unity of the book: the pruning has been severe, perhaps, but at least one will not forget the essentials that have been left. On the other hand, the expository style in which it is written will certainly make the book abominably hard for a beginner, unreasonably hard, I should say. Not because it is unmotivated and dogmatic; as Chevalley points out, it could hardly be otherwise, and anyway, the average bright young graduate students seems to have the universal appetite of a goat. Rather it is simply because throughout the book the helping hand which could point out what is essential in a proof or definition is conspicuously and deliberately withheld. Not until page 223 does the author graciously unbend (perhaps as a reward to the reader for accompanying him so far) and confide that a complicated-looking theorem on the preceding page is only saying that a polynomial in n variables can be looked at as a polynomial in r variables whose coefficients are polynomials in the remaining $n-r$ variables. Some clue as to what is at the back of his mind may be gleaned from the following ringing utterance which ends the preface:

"This is an exercise in rectitude of thought, of which it would be futile to disguise the austerity."

The voice that we hear resounding is that of an Old Testament prophet, but the mental attitude is more like a tenth-grade Latin teacher's, reeking with the old theory of formal disciplines. Thinking rigorously demands first of all a firm grasp of the concepts, otherwise the sort of proof-following which passes for thinking is only a very sophisticated version of computation-checking. It is the difference between a rat running physically through a maze and a man running his pencil through one of the Sunday supplement mazes: one has the over-all understanding, the other does not. It is downright unfair for an older generation which learned these ideas in an intuitive fashion in which they were well-adapted for thought to foist off on a younger one, in the name of rigorous thinking and without any further explanation, such a construction as this one of a free abelian monoid $(\overline{F}, \overline{\Psi})$ on a set of generators S, which occurs at the tender page of 21:

Let N be the additive monoid of integers $\geqq 0$. For any $x \in S$, set $N_x = N$, and form the weak product $\prod_{x\in S}^{w} N_x$. This is a commutative monoid $\overline{F}$. Let ζ_x be the natural injection of N into $\overline{F}$ corresponding to the index x; if $x \in S$, set $\overline{\Psi}(x) = \zeta_x(1)$. Then $\overline{\Psi}$ is a mapping of S into $\overline{F}$. . . etc.

Granted that something like this must be said if one wants to be perfectly precise, nevertheless it obscures completely the *freeness* of the free abelian monoid. If everyone, Chevalley I dare say included, thinks of such a thing as the set of finite unordered formal products of elements of S, why can't this be said in another line or two?

We come finally to the question of the unorthodox subject matter of the book, considered as a first year graduate algebra text. Granting the importance of multilinear algebra, it only follows that it might reasonably be taught in the first year, and not that it necessarily should be. If it is, this means that such things as field theory and much of ring theory (factorizations, Noetherian rings) must go. It is a little hard to wave goodbye—algebra is thereby cut off from number theory and algebraic geometry, its two great classical estuaries. We get to be sure the cohomology theory, but no fields to apply it to.

However, as Chevalley remarks, one cannot teach everything in one year; let us consider multilinear algebra on its own merits. The following remarks are quite frankly subjective. Against this subject as an introduction to algebra, I would argue its dullness. It is a bit difficult to analyze precisely in what this dullness consists. Of course such a theorem as the general associativity law for tensor products is dull, but I mean more than that. Somehow, difficult as some of the material is, one never gets the feeling of advancing in depth; the difficulties lie in keeping track of the ever more complicated piling of module and map, and not as they do in, say, number theory in the inherent intricacy of the God-given structure. We have the feeling of not getting anywhere—we keep studying the same old things about ever more elaborate constructions. We keep on squaring pieces of wood in ever more curious sizes, but we never get to build the table. There is never any *einfall*. One never reads a proof and says "How clever!," one never sees a whole structure revealed by a theorem with a burst of insight—one just keeps plodding along for 200 pages and finally learns what a polynomial really is. Courses and books to accompany them must not do this, especially in early graduate work; they must reach climaxes. There is no Jordan-Hölder theorem, no Galois theorem, no basis theorem for abelian groups, no Wedderburn theorem to be found here—the subject just doesn't contain any. It's a tool subject, and tools should be kept in the closet, no matter how shiny they are.

Just for this reason, Chevalley's book should be welcomed. Now that it has appeared, multilinear algebra need not be taught in courses; the mature student of modern mathematics can learn the subject by himself. He has a fast, well-organized, meaty textbook, and he will be in good shape if he is able to find a friend who will explain now and then what is really going on inside it.

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