THE JUNE MEETING IN PULLMAN

The five hundred thirty-sixth meeting of the American Mathematical Society was held at the State College of Washington in Pullman, Washington, on Saturday, June 15, 1957, preceded by a meeting of the Mathematical Association of America on Friday. There were 97 registrants, including 70 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor M. M. Day delivered an address on *Rotundity and smoothness*. He was introduced by Professor Klee, and the sessions for contributed papers presided over by Professors Paul Civin and J. M. Kingston.

Following are abstracts of papers presented at the meeting, those whose numbers are followed by "*t*" having been given by title. On joint papers, the presenter's name is followed by "(p)". Dr. Schmidt was introduced by Professor Ostrom.

ALGEBRA AND THEORY OF NUMBERS

586. W. E. Barnes and H. Schneider (p): *The group-membership of a polynomial in an element algebraic over a field.*

Let F be a field, R an arbitrary extension ring of F, and let a be an element of R algebraic over F, with minimum polynomial $m(x)$. If $q(x) \in F[x]$, the polynomial $q(a)$ is called a *group-element* in R if there exists a subgroup of the multiplicative semigroup of R to which $q(a)$ belongs. It is proved that $q(a)$ is a group-element in R if and only if the greatest common divisors in $F[x]$ of $q(x)$ and $m(x)$ and of $q(x)^2$ and $m(x)$ are equal, in which case $q(a)$ is a group-element even in $F[a]$. It follows that $q(a)$ is a group-element in R for every polynomial $q(x)$ in $F[x]$ if and only if the irreducible factors of $m(x)$ are simple. This extends a result due to Farahat and Mirsky on the group membership of a matrix polynomial (American Math. Monthly vol. 63 (1956) pp. 410–412). (Received May 1, 1957.)

587. R. A. Beaumont (p) and R. S. Pierce: *Partly transitive modules with proper isomorphic submodules.*

The problem of classifying those R-modules M over a principal ideal domain R which have proper isomorphic submodules is considered. Such modules are called I-modules. For modules of finite rank, a generalization of a theorem of Kaplansky is obtained: Let M be a module of finite rank which splits. Then M is an I-module if and only if M/T, where T is the torsion submodule of M, is not divisible. Using standard techniques from the theory of abelian groups, it is proved that divisible modules and reduced modules may be considered separately, and the divisible case is disposed of easily. Reduced torsion-free modules are I-modules. It is shown that if the cardinality of a minimal set of generators of a primary reduced module M exceeds $\|R\|^{\aleph_0}$, where $\|R\|$ is the cardinality of R, then M is an I-module. Let x and y be elements of a primary module M and let $U(x) = (\alpha_1, \alpha_2, \cdots, \alpha_r, \cdots)$ and $U(y) = (\beta_1, \beta_2, \cdots, \beta_s, \cdots)$ be their Ulm sequences. We write $U(x) \prec U(y)$ if $\alpha_i \leqq \beta_i$ for all i and $\alpha_i = \beta_i$ whenever $\beta_i \geqq \mu$, where μ is the least ordinal such that M_μ, the submodule of elements of height $\geqq \mu$, has finite rank. Then M is called partly transitive if whenever x and y in M satisfy $U(x) \prec U(y)$, then there exists an isomorphism θ of M into itself such that $\theta(x) = y$. It is proved that a countably generated primary module is partly transitive. It follows as a corollary from this result that a primary reduced module with a minimal set of generators of cardinality $\aleph_0$ is an I-module. (Received May 3, 1957.)

588. R. A. Beaumont and R. S. Pierce (p): *Partly invariant submodules of a torsion module.*

Let M be a module over a principal ideal domain R. A submodule N of M is called partly invariant if every isomorphism of M into (not necessarily onto) itself carries N into itself. Of course, any partly invariant submodule is characteristic and any fully invariant submodule is partly invariant. This paper surveys the partly invariant submodules of an arbitrary p-primary torsion module M which is partly transitive (in the sense of the preceding abstract). Let $H = M_\mu$ be the submodule of elements of height $\geqq \mu$ in M, where μ is the least ordinal such that M_μ has finite rank. Let G be a characteristic submodule of H and $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ a finite U-sequence of ordinals $< \mu$ (see Kaplansky's *Infinite Abelian groups*). Then the set of all x in M such that $p^k x \in G$ and $h(p^i x) \geqq \alpha_{i+1}$ for $0 \leqq i < k$ is a partly invariant submodule of M, which is fully invariant if and, assuming M is fully transitive, only if G is fully invariant in H. Moreover, every partly invariant submodule of M, which is not fully invariant, is obtained in this way. These results are of interest only if $R/(p)$ has two elements, since otherwise every characteristic submodule is fully invariant (see Kaplansky, loc. cit.). An example is given of a 2-primary group having characteristic subgroups which are not partly invariant and partly invariant subgroups which are not fully invariant. (Received May 3, 1957.)

589. A. A. Goldstein and Ward Cheney (p): *On overdetermined systems of linear equations*, I.

Let (A_{ij}) be an $n \times (n+1)$ matrix of rank n. Let x minimize $\sum_i (b_i + \sum_j A_{ij} x_j)^2$ where $b \in E_{n+1}$, $x \in E_n$, $i = 1 \cdots n+1$ and $j = 1 \cdots n$. Put $r_i = b_i + \sum_j A_{ij} x_j$. If r has zero components, let them be, say, the first k. If y minimizes $\sum_i (b_i + \sum_j A_{ij} y_j)^2 |r_i|$ subject to the conditions $b_i + \sum_j A_{ij} y_j = 0$ for $i = 1 \cdots k$, then $y = u$ also minimizes $M = \sup_{i \leq n+1} |b_i + \sum_j A_{ij} u_j|$. Put $v_i = (r, r) \operatorname{sgn} r_i \div \sum_s |r_s|$ and let z be chosen so that $b_i + \sum_j A_{ij} z_j = v_i$. Then $z = u$ minimizes M. A vector u which minimizes M is unique if and only if $k = 0$. (Received May 6, 1957.)

590. A. P. Hillman: *A syzygy system in differential algebra.*

Let $y_1, \cdots, y_n$ be indeterminates over a partial differential field $\mathfrak{F}$ of characteristic zero and let $\mathfrak{R} = \mathfrak{F}\{y_1, \cdots, y_n\}$. Let A be in $\mathfrak{R}$ but not in $\mathfrak{F}$. Let S be the separant of A. Let $A_1, \cdots, A_r$ be distinct partial derivatives of A. If $M_1, \cdots, M_r \in \mathfrak{R}$, a necessary and sufficient condition for the syzygy $M_1 A_1 + \cdots + M_r A_r = 0$ is that there exist N_{ij} in $\mathfrak{R}$ and a nonnegative integer t such that $S^t M_i = \sum_{j=1}^r N_{ij} A_j$ for $i = 1, \cdots, r$ and $N_{ij} + N_{ji} = 0$ for $i, j = 1, \cdots, r$. This is generalized in the following: Let $B_0, \cdots, B_r \in \mathfrak{R}$ with B_0 not in $\mathfrak{F}$. Let S be the separant of B_0. Let $\delta_0, \cdots, \delta_s$ be distinct power products in the fundamental differentiations of $\mathfrak{F}$. Then $\sum_{i=0}^g \sum_{j=0}^s M_{ij}(\delta_j B_{g-i}) = 0$ for $g = 0, 1, \cdots, r$ with M_{ij} in $\mathfrak{R}$ if and only if there exist

N_{ijk} in $\mathfrak{R}$ and a nonnegative integer t such that $S^t M_{ij} = \sum_{h=0}^{i} \sum_{k=0}^{s} N_{hjk}(\delta_k B_{i-h})$ and $N_{ijk} + N_{ikj} = 0$ for $i = 0, \cdots, r$ and $j, k = 0, \cdots, s$. (Received May 1, 1957.)

591t. J. H. Hodges: *Some matrix equations over a finite field.*

Let $GF(q)$ denote the finite field of $q = p^n$ elements, p odd. In this paper we consider several problems of the following type. Let B be a symmetric matrix of order t and A be an arbitrary $m \times t$ matrix, both with elements in $GF(q)$. Determine the number of $t \times m$ matrices X over $GF(q)$ such that $X'A + A'X = B$. If the equation has any solutions, it is shown that their number is q^e, where $e = t(m-r) + r(r-1)/2$ and r is the rank of A. A necessary and sufficient condition is given for the existence of solutions of the equation. (Received April 30, 1957.)

592t. J. H. Hodges: *On orthogonal matrices of polynomials over a finite field.*

J. L. Brenner (*Polynomial parametrizations*, Amer. Math. Monthly vol. 58 (1951) pp. 327–329) has proved that the only orthogonal matrices of order 2 with elements in $GF[p, x]$ are constant matrices if $p > 2$ and if $p = 2$ are of the form $A = I + p(x)C$, where C is the matrix of order two all of whose elements are $+1$ and $p(x)$ is an arbitrary element of $GF[p, x]$. L. Carlitz (*A note on orthogonal matrices*, Amer. Math. Monthly vol. 60 (1953) pp. 253–255) has given examples of nonconstant orthogonal matrices of order p over $GF[p, x]$ for $p > 2$. In (*Orthogonal matrices of modular polynomials*, Duke Math. J. vol. 21 (1954) pp. 225–232) Brenner gives a construction for producing nonconstant orthogonal matrices of order 3, therefore of arbitrary order $m > 2$, over $GF[p, x]$ for $p > 2$. In the present paper it is shown that nonconstant orthogonal matrices of order $m > 2$ with elements in $GF[q, x]$, $q = p^n$ odd, may be constructed using results by Eckford Cohen concerning sums of squares in $GF[q, x]$ (*Sums of an even number of squares in* $GF[p^n, x]$, II, Duke Math. J. vol. 14 (1947) pp. 543–557 and *Sums of an odd number of squares in* $GF[p^n, x]$, Duke Math. J. vol. **15** (1948) pp. 501–511). (Received April 30, 1957.)

593t. Joseph Landin and Irving Reiner: *Automorphisms of the two-dimensional general linear group over a euclidean ring.*

Let R be a commutative principal ideal domain which is integrally closed in its quotient field, and assume (i) R is euclidean, (ii) the group U of units of R contains more than two elements, (iii) every element of R can be expressed as a finite linear combination of units, with coefficients which are rational integers (when R has characteristic 0) or elements of $GF(p)$ (when R has characteristic $p > 0$). It is shown that every automorphism of $GL_2(R)$ can be expressed as a product of automorphisms of the following types: (i) inner, (ii) $u \to u^\sigma$, where σ is an automorphism of the ring R, (iii) $u \to \lambda$ (det u)u, where λ is a homomorphism of U into itself satisfying the condition: $\lambda(\alpha^2) = \alpha^{-1}$ if and only if $\alpha = 1$. (Received March 14, 1957.)

594t. Howard Osborn: *A class of bilinear forms.* I.

Let E be a finite-dimensional vector space over an arbitrary field, A an endomorphism of E, and $\mathfrak{A}$ the ring generated by A and scalar multiplications. As an $\mathfrak{A}$-module E is the direct sum $E_1 \oplus \cdots \oplus E_t$ of unique cyclic $\mathfrak{A}$-submodules such that the order ideal of E_i contains that of E_t when $i \leqq j$. For any bilinear form f on $E \times E$ let fA represent that form such that $fA(u, v) = f(u, Av)$. If g is a given form on $E \times E$ let $\mathcal{S}(A; g)$ be that class of symmetric forms f such that $fA + g$ is also symmetric, and let $\mathfrak{A}(A;g)$ be that class of alternating forms f such that $fA+g$ is also alternating. *THEOREM: Let x_j' be an arbitrary linear functional on $E_1\oplus\cdots\oplus E_j$, and let z_j generate E_j. Then there exists a unique $f\in\mathfrak{S}(A;g)$ such that $x_j'(x_j)=f(x_j,z_j)$ for all $x_j\in E_1\oplus\cdots\oplus E_j$, $j=1,\cdots,t$, and if $t>1$ there exists a unique $f\in\mathfrak{A}(A;g)$ such that $x_{j-1}'(x_{j-1})=f(x_{j-1},z_{j-1})$ for all $x_{j-1}\in E_1\oplus\cdots\oplus E_{j-1}$, $j=2,\cdots,t$.* (Received May 1, 1957.)

595t. Howard Osborn: *A class of bilinear forms. II.*

Let $\mathfrak{C}(A)$ represent the linear space of bilinear forms f such that A is self-adjoint with respect to f, $f(Au,v)=f(u,Av)$, and let $\mathfrak{E}(A)$ represent the centralizer of A in the ring of all endomorphisms of E. Then $\mathfrak{C}(A)$ and $\mathfrak{E}(A)$ are isomorphic right $\mathfrak{E}(A)$-modules, where $f\cdot B=fB$ for $B\in\mathfrak{E}(A)$, and a fortiori $\mathfrak{C}(A)$ and $\mathfrak{E}(A)$ are isomorphic $\mathfrak{A}$-modules. Furthermore, for fields of characteristic $\neq 2$, $\mathfrak{S}(A;0)$ and $\mathfrak{A}(A;0)$ are $\mathfrak{A}$-modules such that $\mathfrak{C}(A)=\mathfrak{S}(A;0)\oplus\mathfrak{A}(A;0)$. This leads to a development of $\mathfrak{E}(A)$ as an $\mathfrak{A}$-module, from which one obtains an easy characterization of $\mathfrak{E}(A)$ as an algebra over $\mathfrak{A}$. In particular $\mathfrak{C}(A)$ is the direct sum over $\mathfrak{A}$ of a special Jordan algebra and a Lie algebra, these two algebras corresponding to $\mathfrak{S}(A;0)$ and $\mathfrak{A}(A;0)$, respectively. (Received May 1, 1957.)

596t. R. M. Robinson: *The converse of Fermat's theorem.*

It is shown that a theorem of Lehmer concerning the form of the prime factors of a composite number satisfying Fermat's congruence is in a certain sense the strongest possible result. Particular attention is given to the use of Euler's criterion as a test for primeness. The discussion here is based on the following theorem: *Let $N=k\cdot 2^n+1$, where $k>0$, $n>0$, and k is odd. Then there exists a such that $a^{(N-1)/2}\equiv -1$ (mod N) if and only if every prime factor p of N satisfies the congruence $p\equiv 1$ (mod 2^n).* As a by-product of this study, the following rather unexpected result is obtained: *If $N>1$ is odd, and $a^{(N-1)/2}\equiv -1$ (mod N), then $(a/N)=-1$.* (Received April 30, 1957.)

597t. H. J. Ryser: *The term rank of a matrix.*

This paper continues the author's study of the combinatorial properties of a matrix A of m rows and n columns, all of whose entries are 0's and 1's (Bull. Amer. Math. Soc. Abstract 62-6-636). Let $\mathfrak{A}$ be the class of 0, 1 matrices A of m rows and n columns, with row sum vector $R=(r_1,\cdots,r_m)$ and column sum vector $S=(s_1,\cdots,s_n)$. Let the components of R and S be positive, and let $R^1=(r_1-1,\cdots,r_m-1)$. Let $\overline{A}'$ be the 0, 1 matrix of size m by n with row sum vector R' and such that the 1's in each row of $\overline{A}'$ are shifted to the left as far as possible. Let the column sum vector of $\overline{A}'$ equal $\overline{S}'=(\bar{s}_1',\cdots,\bar{s}_n')$. Renumber the subscripts of S so that $s_1\geqq\cdots\geqq s_n$, and define $s_i'=s_i-1$, $i=1,\cdots,n$. Let $\bar{s}_0'=s_0'=0$. Let M be the largest integer in the set $\sum_{i=0}^{k}(s_i'-\bar{s}_i')$, $k=0,\cdots,n$. Then if $\bar{\rho}$ is the maximal term rank for the matrices in $\mathfrak{A}$, $=m-M$. This result is applied to a variety of combinatorial problems dealing with term ranks of 0, 1 matrices in the class $\mathfrak{A}$. (Received March 8, 1957.)

598. Wolfgang Schmidt: *The measure of the set of admissible lattices.*

If S is a point set in R_n, then, as usual, we say a point lattice is admissible relative to S, if it has no point in S, except possibly the origin. Let $A(S)$ be the set of admissible lattices of determinant 1. We prove the theorem: Suppose S is a Borel set with measure $V=V(S)\leqq c_1n$. Then $m(A(S))=\int_{A(S)}d\mu=e^{-V}(1-R)$, where $|R|<\exp(2V-2c_2n)$.

Here $d\mu$ is the invariant measure over the space of lattices of determinant 1, and $c_1>c_2>0$ are constants. The proof is based on results of C. A. Rogers on mean values over lattices. A consequence of the above theorem is an improvement of the Minkowski-Hlawka theorem: If $V(S)\leqq c_2 n$, then there exist admissible lattices of determinant 1. (Received April 29, 1957.)

599t. S. K. Stein: *The union of arithmetic sequences.*

Theorem 1. If A_i, $1\leqq i\leqq n$, and B_j, $1\leqq j\leqq m$ are each collections of intersecting incomparable arithmetic sequences and $\bigcup A_i=\bigcup B_j$ then $m=n$ and the B_j coincide (in some order) with the A_i. *Theorem* 2. If A_i, $1\leqq i\leqq n$ and B_j, $1\leqq j\leqq m$ are each collections of disjoint incongruent arithmetic sequences and $\bigcup A_i=\bigcup B_j$ then $m=n$ and the B_j coincide (in some order) with the A_i. The first theorem is easily proved by the Chinese Remainder Theorem, the second, by translation of the problem into one on the factorization of polynomials with complex coefficients. Theorem 2 yields a new proof of Mirsky-Newman's theorem (given in P. Erdös *On a problem concerning congruence systems* (In Hungarian)) which asserts that the integers J are not the finite union of disjoint incongruent arithmetic sequences (unless the collection is simply J itself). *Theorem* 3. The complement of a finite union of disjoint incongruent arithmetic sequences of differences $d_1<d_2<\cdots<d_n$ is again an arithmetic sequence if and only if $d_i=2^{-i}$. The proof is similar to that of Theorem 2. (Received April 18, 1957.)

600t. Robert Steinberg: *Groups generated by reflections.*

H. S. M. Coxeter [see *Regular polytopes*, London, 1948, and New York, 1949, pp. 212, 227] has proved by verification two theorems ((1) and (2) below) concerning finite orthogonal groups generated by reflections in real finite-dimensional Euclidean spaces. In this paper, general proofs are given for these theorems and some related results. (1) Let G be a finite irreducible g.g.r., let $W_1, W_2, \cdots, W_n$ be the walls of a fundamental region and $R_1, R_2, \cdots, R_n$ the reflections in these walls. Then, if $R_1R_2\cdots R_n$ has order h, the number of reflecting hyperplanes is $nh/2$. The method of proof leads to an explicit enumeration of the reflecting hyperplanes in terms of the W's and R's. In the case that G is associated with a simple Lie group [see H. Weyl, *Continuous groups*, I.A.S. notes], an enumeration of the positive roots in terms of the fundamental set results, and this is used to prove (2): if $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a fundamental positive set of roots and if $\sum z^j\alpha_j$ is the dominant root, then $\sum z^j=h-1$. (Received May 4, 1957.)

Analysis

601t. G. L. Krabbe: *Spectral invariance of convolution operators on* $L^p(-\infty, \infty)$.

Suppose that g is a function of bounded variation on $(-\infty, \infty)$ satisfying the condition $g(-\infty)=0$, and let $\mathfrak{M}^1$ be the set of all such functions. Denote by g_{*p} the mapping $f\rightarrow g^*f$ defined for all f in L^p (as usual, $(g^*f)(\theta)=\int f(\theta-\alpha)dg(\alpha)$). This note concerns the spectrum $\sigma(g_{*p})$ of the operator g_{*p}; let $\mathfrak{F}$ be the set of all the members g of $\mathfrak{M}^1$ such that $\sigma(g^*_p)=\sigma(g_{*1})$ for all $p>1$. Theorem 1: $g\in\mathfrak{F}$ if the Fourier transform Fg of g is of the form $\sum a_n \exp(2\pi i\theta\alpha_n)$ with $\sum|a_n|<\infty$ $(n=-\infty, \cdots, \infty)$; note that in this case $g_{*p}=\sum a_n t(\alpha_n)$, where $t(\alpha_n)$ is the translation operator defined by $(t(\alpha_n)f)(\theta)=f(\theta+\alpha_n)$. Theorem 2: $\mathfrak{F}$ contains all absolutely continuous functions.

Theorem 3: If $g \in \mathfrak{F}$, then $\sigma(g_{*p})$ is the closure $r(Fg)$ of the range of $\mathfrak{F}g$ (for any $p \geqq 1$). The theorems 2 and 3 combine to form a generalization of a Hilbert space result due to H. Pollard [*Integral transforms*, Duke Math. J. vol. 13 (1946) pp. 307–330] (see also N. Dunford's extension in the Proc. Symposium Spectral Theory, Oklahoma, 1955, pp. 57–62). Since $\mathfrak{F}$ is also the set of all g in $\mathfrak{M}^1$ with $\sigma(g) \subset r(\mathbf{F}g)$, it follows immediately from results of N. Wiener and Pitt that $\mathfrak{F} \neq \mathfrak{M}^1$. We characterize a subset $\mathfrak{F}_2$ of $\mathfrak{F}$ which includes the sets involved in the theorems 1 and 2; it seems probable that $\mathfrak{F}_2 = \mathfrak{F}$. (Received April 17, 1957.)

602*t*. Marvin Rosenblum: *On the Hilbert matrix*, I.

1. For fixed $k<1$ the generalized Hilbert matrix is $H_k = ((m+n+1-k)^{-1})$, $m, n = 0, 1, 2, \cdots$. By an *eigenvalue* of H_k we mean a complex number λ such that there exists a non-null sequence of complex number $\{x_n\}_0^\infty$ with the property that $\sum_{n=0}^\infty (n+m+1-k)^{-1} x_n$ converges to λx_m for all non-negative integers m. Theorem. Every complex number with positive real part is an eigenvalue of H_k. (Received April 11, 1957.)

603. R. G. Selfridge: *Summability of Walsh transforms.*

The Walsh functions form a complete system over the unit interval, and one can generate Walsh-Fourier series. A transform can also be generated for functions in $L_p(0, \infty)$ which has an inverse which converges to the initial function under reasonable conditions. The inverse transform is also C, 1 summable to the initial function under certain conditions. Previously C, 1 summability had not been proven almost everywhere. This paper, based primarily on a paper by N. J. Fine (Proc. N. A. S. vol. 41 (1955) pp. 588–591) shows that the inverse transform of the transform of a function in $L(0, \infty)$ is summable almost everywhere to the initial function. The same result also holds for functions in $L_p(0, \infty)$ for $1<p\leqq 2$. (Received March 18, 1957.)

604*t*. V. L. Shapiro: *Uniqueness of surface spherical harmonic series.*

Let $f(x)$ be in L^1 on S, the surface of the unit sphere in three dimensional Euclidean space. If $f(x)$ has for its Laplace series $\sum_{n=0}^\infty Y_n(x)$, then $\|Y_n\| = o(n^{1/2})$ where $Y_n(x)$ is a surface spherical harmonic of order n and $\|Y_n\|^2 = \int_S |Y_n(x)|^2 dS(x)$, $dS(x)$ being the natural two dimensional area element on S. In this paper we prove three theorems concerning arbitrary series of surface spherical harmonics which satisfy the condition $\|Y_n\| = o(n^{1/2})$. *Theorem* 1. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let $f^*(x)$ and $f_*(x)$ be respectively the upper and lower Abel limits of the series. Suppose (i) $\|Y_n\| = o(n^{1/2})$, (ii) $f_*(x)$ and $f^*(x)$ are finite for x on S, (iii) $f_*(x) \geqq A(x)$ where $A(x)$ is in L^1 on S. Then $f_*(x)$ is in L^1 on S and the given series is its Laplace Series. *Theorem* 2. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let p be a fixed point on S. Suppose (i) $\|Y_n\| = o(n^{1/2})$, (ii) $f_*(x)$ and $f^*(x)$ are finite on $S-p$, (iii) $f_*(x)$ and $f^*(x)$ are in L^1 on S. Then the given series is the Laplace series of $f_*(x)$. *Theorem* 3. Given the series of surface spherical harmonics, $\sum_{n=0}^\infty Y_n(x)$. Let E be a denumerable point set on S. Suppose (i) $\|Y_n\| = o(n^{1/2})$ (ii) $f_*(x) = f^*(x) = 0$ for x in $S-E$. Then $Y_n(x) = 0$ for all n. It is easy to see that Theorems 2 and 3 are best possible results. (Received February 27, 1957.)

605. Abraham Spitzbart and Nathaniel Macon (p): *Numerical differentiation formulas.*

In a recent paper (Amer. Math. Monthly vol. 64 (1957) pp. 79–82) R. T. Gregory described an algorithm for finding the coefficients in certain formulas which express the derivatives of an interpolation polynomial at a set of equally spaced points as linear combinations of the given functional values. The method requires the solution of systems of linear equations or, equivalently, the inversion of certain Vandermonde matrices. In this paper, explicit representations for these same coefficients are obtained, and in such a form that their use is in no way restricted to the tabular points. These representations are in terms of Stirling numbers, which are well known and extensively tabulated. This in turn enables one to determine directly the matrix inverses mentioned above. Applications to more general matrices and general interpolation problems will be made later. (Received April 18, 1957.)

Applied Mathematics

606. V. W. Bolie: *Electromagnetic propagation in an almost homogeneous medium.*

This paper concerns the developnent from Maxwell's electromagnetic equations an equation of propagation in an almost homogeneous medium. The equation is applied to the problem of determining the secondary wave produced by an isolated gaussian-shaped perturbation in the refractive index. An exact solution is obtained for points located on the axis of symmetry parallel to the direction of propagation of the incident wave. An approximate solution for points remote from the anomaly is obtained and its validity is compared with the more restricted exact solution. An interesting limit process is encountered in the derivation of the formula for the scattering cross section of the refractive index perturbation. (Received April 22, 1957.)

607*t*. Philip Wolfe: *Simplex method for quadratic programming.*

Given matrices: A, m by n; b, m by 1, $b \geqq 0$; C, m by n, positive semidefinite. To solve: Minimize $x'Cx$ subject to $x \geqq 0$, $Ax = b$. Let $'$ denote transposition, $x = (x_1, \cdots, x_n)'$, $v = (v_1, \cdots, v_n)$, $u = (u_1, \cdots, u_m)$, $z = (z_1, \cdots, z_n)'$. *Initiation*: Let $Ax^0 = b$. Choose $z^0 \geqq 0$ and the diagonal matrix E so that $|E_{ii}| = 1$ and $Cx^0 + Ez^0 = 0$. Let $v^0 = 0$, $u^0 = 0$. *Recursion*: Given the basic feasible vector $[x^k, v^k, u^k, z^k]$ satisfying the constraints below, take one step in the simplex method minimization of $\sum_j z_j$ subject to $x \geqq 0$, $v \geqq 0$, $z \geqq 0$, $Ax = b$, $x'C + uA - v + Ez = 0$ with the added condition: Do not allow $x_j^{k+1} > 0$ if $v_j^k > 0$, or $v_j^{k+1} > 0$ if $x_j^k > 0$. *Theorem*: The procedure will terminate in at most $C_{m+3n,m+n}$ steps with $z = 0$; the final x solves the minimization problem. (Received April 15, 1957.)

Geometry

608. T. G. Ostrom: *Dual transitivity in finite projective planes.*

The author defines a plane π to be dually transitive if, for every pair of points P and P_1, and every pair of lines l and l_1 (where P is not on l and P_1 is not on l_1) there is a collineation of π which maps P into P_1 and l into l_1. Dual transitivity will follow if, for every such pairs of points and lines, there is a duality which maps P into l_1 and l into P_1. The main result is that if the plane is finite of order n, where n is not a square, then the plane is Desarguesian. See the author's paper *Double transitivity in finite projective planes*, Canadian J. Math. vol. 8 (1956) pp. 563–567. (Received April 29, 1957.)

609. F. A. Valentine: *The intersection of two convex surfaces and property* P_3.

A set S in a linear space L has property P_3 if for each triple of points x, y, z in S at least one of the segments xy, yz, xz is in S. This property enables one to establish the following: *Let S_1 and S_2 be two compact convex bodies in a finite dimensional normed linear space L_n, and suppose their interiors have a nonempty intersection. Let $B(S_i)$ denote the boundary of S_i ($i=1, 2$). If $B(S_1)\cdot B(S_2)$ is in the interior of the convex hull of $S_1\cup S_2$, then $B(S_1)\cdot B(S_2)$ is the union of a finite number of disjoint compact ($n-2$)-dimensional manifolds.* Furthermore: *Let S be a closed set in a topological linear space L, where dimension $L>2$. Assume S has property* P_3, *and that S is not contained in any two-dimensional variety of L. If the set Q of points of local nonconvexity of S has an isolated point, then Q has at most two points* (a point p is a point of local convexity of S if there exists a neighborhood N of p such that for each pair of points x and y in $S\cdot N$, it is true that $xy\subset S$). (Received April 22, 1957.)

LOGIC AND FOUNDATIONS

610*t*. J. C. E. Dekker: *On certain equations in isols*, II.

For notations and terminology see the author's two abstracts, J. Symbolic Logic vol. 20 (1955) pp. 204, 205. The relation $(\exists U)[A+U=B]$ is denoted by $A\leqq B$ or $B\geqq A$. An abstract of the first part of the present paper appears in the March, 1957 issue of the J. Symbolic Logic. In the following C and k stand for arbitrary constants in Λ, C finite or infinite, but k finite. The letters X, Y, U, V are used as variables ranging over Λ. The function $X(X+1)\cdots(X+k)$ is denoted by $\Pi(X, k)$. *Theorems.* (1) $XC=YC$ and $C\geqq 1$ imply $X=Y$, (2) $X^C=Y^C$ and $C\geqq 1$ imply $X=Y$, (3) $C^X=C^Y$ and $C\geqq 2$ imply $X=Y$, (4) $\Pi(X, k)=\Pi(Y, k)$ implies $X=Y$, (5) $X^X=Y^Y$ and X, $Y\geqq 1$ imply $X=Y$, (6) $X^2+Y^2=2XY$ implies $X=Y$, (7) $j(X, Y)=j(U, V)$ implies $X=U$ and $Y=V$. It follows from (6) that the ring Λ^* has no nilpotents. While the function $j(X, Y)$ maps $\epsilon\times\epsilon$ onto ϵ, it maps $\Lambda\times\Lambda$ into Λ, but not onto Λ. (Received April 25, 1957.)

611*t*. J. C. E. Dekker: *Congruences in isols with a finite modulus.*

For notations and terminology see the preceding abstract. Let m denote a positive integer. We say that *m divides X* (written: $m|X$) if $(\exists U)[mU=X]$. If m and X have no common divisor besides 1, they are called *relatively prime*. $X\equiv Y \pmod m$ means: $X\geqq Y$ and $m|X-Y$. It is easily seen that: if $X_1\equiv Y_1 \pmod m$ and $X_2\equiv Y_2 \pmod m$, then $X_1+X_2\equiv Y_1+Y_2 \pmod m$ and $X_1X_2\equiv Y_1Y_2 \pmod m$. On the other hand, the three conditions $ZX\equiv ZY \pmod m$, $Z\neq 0$, Z and m relatively prime do not imply $X\equiv Y \pmod m$. *Theorem.* Let p be any prime. Then (1) $(\forall X)[X\not\equiv 0 \pmod p\rightarrow X^{p-1}\equiv 1 \pmod p]$ is false, but (2) $(\forall X)[X^p\equiv X \pmod p]$ is true. *Corollary.* Let p be any prime. Then (1) $p|XY$ does not imply $p|X$ or $p|Y$, but (2) for $n\geqq 1$, $p|X\leftrightarrow p|X^n$. *Indication of proof.* The isol X is called *indecomposable*, if it is not the sum of two infinite isols. The first part of the theorem follows from the fact (due to Myhill) that there exist infinite, indecomposable isols. The second part can be established directly by a modification of the proof of $(\forall x)[x^p\equiv x \pmod p]$ due to M. J. Perott [Bull. des Sc. Math. vol. 24 (1900) pp. 175, 176] or indirectly as a consequence of $\Pi(X, p-1)\equiv 0 \pmod{p!}$. (Received April 25, 1957.)

STATISTICS AND PROBABILITY

612. D. G. Chapman: *A comparative study of several one-sided goodness-of-fit tests.*

Let X be a real random variable with d.f. $F \in \Omega_2$, the class of continuous distribution functions on R. Several strongly distribution-free tests of the hypothesis H_0: $F = F_0$ (where F_0 is completely specified) against the alternative $F < F_0$ have been proposed. These tests are reviewed in the light of criteria that have been suggested as desirable for tests in general and in the light of two new criteria: partial ordering and monotonicity. Further let $\rho^-(F_0, F) = \sup_{-\infty<x<\infty} [F_0(x) - F(x)]$. Sharp upper and lower bounds are found for those tests of H_0 which meet the specified criteria, against alternatives F such that $\rho^-(F_0, F) = \Delta$. (Received April 26, 1957.)

613. R. K. Getoor: *Conditional expectations of L^* random variables.*

Let $(\Omega, \mathfrak{F}, P)$ be a probability space then Mourier (Proc. 3rd Berk. Symp. on Math. Stat. and Prob. vol. II) has defined an L^* random variable to be a mapping, X^*, from Ω into the dual, $\mathfrak{X}^*$, of a separable Banach space, $\mathfrak{X}$, which is weak* measurable, i.e., $\langle x, X^*(\omega)\rangle$ is $\mathfrak{F}$-measurable for every $x \in \mathfrak{X}$. Let $\mathfrak{F}'$ be a σ-subalgebra of $\mathfrak{F}$ and suppose that $E(\|X^*\|) < \infty$, then there exists an L^* random variable, $E(X^* | \mathfrak{F}')$, which is weak* measurable relative to $\mathfrak{F}'$ and satisfying $\int_\Lambda X^* dP = \int_\Lambda E(X^* | \mathfrak{F}') dP$ for every $\Lambda \in \mathfrak{F}'$. $E(X^* | \mathfrak{F}')$ will be called the conditional expectation of X^* given $\mathfrak{F}'$. The elementary properties of $E(X^* | \mathfrak{F}')$ are derived and martingales of L^* random variables are investigated. (Received April 10, 1957.)

614. H. G. Tucker: *Limit theorems for interpolated random variables.* Preliminary Report.

Sometimes it is difficult, impossible, or costly to observe all of n independent, identically distributed random variables or even to observe the class intervals of a histogram into which all n random variables fall. The case considered here is that in which the number of random variables which fall into alternate class intervals can be counted; the number of random variables which fall into intervals not observed are then interpolated. It is assumed that the common distribution function $F(x)$ possesses a density which is bounded and continuous almost everywhere. By means of this partly observed, partly manufactured histogram, an estimate $F^*(x)$ of $F(x)$ is constructed. By taking limits in an appropriate way, the following limit theorems are proved: (i) $F^*(x)$ converges to $F(x)$ uniformly in x with probability one, (ii) if $\int x dF(x)$ exists and is finite, then $\int x dF^*(x)$ converges to $\int x dF(x)$ with probability one, and (iii) if $\int x^2 dF(x)$ is finite, then $\int x dF^*(x)$ is asymptotically normal. (Received April 29, 1957.)

615t. Jacob Wolfowitz: *The coding of messages subject to chance errors.*

A "channel probability function" p with memory m is defined on the set of all sequences of $(m+1)$ zeros or ones; $0 \leq p \leq 1$. Theorem 1: Let $X_1, X_2, \cdots$ be a stationary, metrically transitive Markov chain with two states, zero and one, and let $Y_1, Y_2, \cdots$ be a sequence of chance variables such that Y_i is 0 or 1 and $P\{Y_i = 1 | X_1, X_2, \cdots, Y_1, \cdots, Y_{i-1}\} = p(X_i, X_{i+1}, \cdots, X_{i+m})$. Let $H(Y)$ be the entropy of the Y-process, $H_X(Y)$ be its entropy relative to the X-process, and $\lambda > 0$.

There is a $K>0$ and, for any n, an error correcting code of length $2^{n(H(Y)-H_X(Y))-Kn^{1/2}}$. The probability that any message (of n symbols) will be incorrectly received is $<\lambda$. Theorem 2. Let $m=0$ and C_0 be the capacity of the channel. There exists a $K'>0$ such that, for any n, a code such that the probability of transmitting any word incorrectly is $<\lambda$, has length less than $2^{nC_0+K'n^{1/2}}$. These results generalize immediately to alphabets with a finite number of symbols. (Received March 4, 1957.)

Topology

616*t*. S. K. Stein: *Continuous choice functions and convexity.*

Theorem. An open set A in E_n (n dimensional Euclidean space) is convex iff there exists a continuous choice function on the space of r-dimensional cross-sections of A, $1 \leq r \leq n-1$. The proof is first provided for the case $n-r=1$ by topological arguments and then proceeds by induction on $n-r$. *Theorem.* An open set A in E^n is convex iff there exists a continuous faithful choice function on the space of caps of A. The proof of this uses the fact that if ϕ: $S^{n-1} \rightarrow S^{n-1}$($S^{n-1}=n-1$ dimensional sphere) is not the identity but ϕ^2 is the identity and if f: $S^{n-1} \rightarrow S^{n-1}$ satisfies $f\phi = f$ then f has even degree. These theorems imply and are suggested by some of the work of Peano, Ascoli, Dupin, Krafft, and Steinhaus on centers of gravity. (Received April 18, 1957.)

617*t*. L. B. Treybig: *Concerning certain locally peripherally separable spaces.*

In this paper on a property related to separability in metric spaces, Journal of the Elisha Mitchell Scientific Society, vol. 70, no. 1, June, 1954, F. Burton Jones indicated that the question whether or not every connected, locally peripherally separable, metric space is separable seems to remain unanswered. In the present paper the author answers the question in the negative. (Received April 4, 1957.)

V. L. Klee, Jr.,
Associate Secretary