

## THE APRIL MEETING IN MONTEREY

The five hundred twenty-fifth meeting of the American Mathematical Society was held at the U. S. Naval Postgraduate School in Monterey, California, on Saturday, April 28, 1956. Attendance was approximately 140, including about 110 members of the Society.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor H. C. Wang delivered an address on *Some aspects of transformation groups and homogeneous spaces*. He was introduced by Professor Z. W. Birnbaum. Presiding at the sessions for contributed papers were Professors Paul Garabedian, Ivan Niven, and Raphael Robinson.

Following are the abstracts of papers presented at the meeting, those whose numbers are followed by "t" having been given by title.

Where a paper has more than one author, the paper was presented by the author whose name is followed by "(p)". Mrs. Butler was introduced by Professor Alfred Tarski, Mr. Hanf by Professor Bjarni Jonsson, and Professor Craig by Dr. R. L. Vaught.

### ALGEBRA AND THEORY OF NUMBERS

552. Jean W. Butler: *On operations in finite algebras.*

Consider a finite set  $A$  with  $n \geq 2$  elements. Let  $F_A^m$  be the set of all functions ( $m$ -ary operations) on  $A \times \cdots \times A$  ( $m$  times) to  $A$ ;  $F_A$  be the union  $F_A^1 \cup F_A^2 \cup \cdots \cup F_A^m \cup \cdots$ . For  $X \subseteq F_A$ ,  $\bar{X}$  denotes the smallest  $Y \subseteq F_A$  with the properties:  $X \subseteq Y$ ; if  $f \in Y$  and  $h$  is obtained from  $f$  by identifying or transposing two arguments, or by substituting a function  $g \in Y$  for an argument, then  $h \in Y$ .  $X$  is closed iff  $\bar{X} = X \subseteq F_A$ ;  $Y$  is a basis of  $X$  iff  $\bar{Y} = X$  and  $\bar{Z} \neq X$  whenever  $Z \subset Y$ .  $F_A$  has a finite basis (Post, 1921). Theorems: (I) For every closed  $X \subseteq F_A^1$  there is a largest closed  $Y \subset F_A$  such that  $Y \cap F_A^1 = X$ . (II) There exist  $p$  closed sets  $M_1, \dots, M_p \subset F_A$  ( $p$  finite, depending on  $n$ ) such that every closed  $X \subset F_A$  is included in some  $M_i$ . (III) There is an integer  $p$  (depending on  $n$ ) such that every basis of  $F_A$  has  $\leq p$  elements. (II) follows from (I), and (III) from (II). (I)–(III) generalize results of Post (Annals of Mathematics Studies, No. 5) for  $n=2$ . Tarski has noticed that a modification in the proofs of (II) and (III) leads to more general results (II') and (III') differing from (II) and (III) only in that  $F_A$  is replaced by any closed set  $Z$  with a finite basis. (Received April 25, 1956.)

553t. Anne C. Davis: *On simply ordered relations with nontrivial automorphism groups.*

Let  $S$  be a simply ordered relation, let  $G(S)$  be the group of automorphisms of  $S$  and let  $\alpha$  be the order type of  $S$ . Theorem 1.  $G(S)$  consists of more than one element if and only if  $\alpha$  is representable in the form  $(1) \alpha = \beta + \gamma \cdot (\omega^* + \omega) + \delta$ , where  $\gamma \neq 0$ . Lemma 2. Suppose that  $f, g \in G(S)$  and  $x$  belongs to the field of  $S$ . Let  $\beta$  be the type of the subrelation of  $S$  whose field consists of all elements  $h(x)$ , where  $h$  belongs to

the subgroup of  $G(S)$  generated by  $f$  and  $g$ . Then either (i)  $\beta = (\omega^* + \omega)^m \cdot \eta^n$ , with  $m$  a non-negative integer and  $n \in \{0, 1\}$ , or (ii) there exists a sequence  $\delta_0 \cdots \delta_n \cdots$  such that, for each  $n$ ,  $\beta = (\omega^* + \omega)^n \cdot \delta n$ . (Lemma 2 is due in part to a suggestion of Tarski.) Theorem 3. If  $S'$  is a scattered relation of type  $\alpha'$ , then  $G(S')$  is non-Abelian if and only if  $\alpha'$  is representable in the form  $\alpha' = \beta' + \alpha \cdot (\omega^* + \omega) + \delta'$ , with  $\alpha$  satisfying (1). Theorem 1 answers Problem (a) of Bull. Amer. Math. Soc. Research Problem 60-3-10, proposed by Goffman. Theorem 3 gives a partial answer to Problem (b) mentioned therein, but in the general case that problem remains open. (Received March 6 1956.)

554. William Hanf: *Representations of lattices by subalgebras*. Preliminary report.

Following Birkhoff and Frink [Trans. Amer. Math. Soc. vol. 64 (1948) pp. 29-316] we call an element  $a$  of a complete lattice  $L$  inaccessible if  $a \in K$  whenever  $K$  is a directed set such that  $\sum_{y \in K} y = a$ . Given an algebra  $A$  with countably many operations of finite rank, the lattice  $L$  of all subalgebras of  $A$  has the following properties: (i)  $L$  is complete, (ii) every element of  $L$  is a sum of inaccessible elements, (iii)  $x \sum_{y \in K} y = \sum_{y \in K} xy$  for any directed set  $K$  and (iv) the set of all inaccessible elements less than a given inaccessible element is countable. Conversely, every lattice  $L$  which satisfies (i)-(iv) is isomorphic to the lattice of all subalgebras of an algebra  $A$  with countably many operations of finite rank—in fact,  $A$  can be taken to be a commutative loop. This supplements the result of Birkhoff and Frink [ibid. Theorem 2] that conditions (i)-(iii) are necessary and sufficient for a lattice to be isomorphic to the lattice of all subalgebras of an algebra with arbitrarily many operations of finite rank. The problem was suggested by B. Jónsson. (Received March 5, 1956.)

555. Arno Jaeger: *Multidifferentiations of finite dimension in commutative rings of nonzero characteristic*. Preliminary report.

The newly developed theory of differentiations of commutative rings of square-free nonzero characteristic, having an identity and inverses to all nonzerodivisors, is generalized to multidifferentiations of finite dimension (cf. J. Reine Angew. Math. vol. 190 (1952) pp. 1-21). The concept of regularity plays the central role. Regular multidifferentiations can be approximated by systems of partial differentiations. The classes of chain-rule-dependent multidifferentiations are investigated. An addition is defined for commuting multidifferentiations of equal dimension, and all multidifferentiations of a class are exhibited. If there do not exist regular multidifferentiations of arbitrarily high dimension in a ring then each differentiation is shown to be dependent on any regular multidifferentiation of maximal dimension. (Received March 5, 1956.)

556t. S. A. Jennings and Rimhak Ree. *Derivations of the group algebras of a class of  $p$ -groups*.

Let  $G$  be a  $p$ -group of class 2 with  $A^p = 1$  for all  $A \in G$ , let  $Z$  be the center and  $C$  be the derived group of  $G$ . If  $\Gamma = \Gamma(G)$  is the group algebra of  $G$  over a field  $\Phi$  of characteristic  $p$ , we may consider the Lie algebra  $L$  of all derivations of  $\Gamma$  over  $\Phi$ . We prove that  $L$  contains a solvable ideal  $S$  such that  $L/S \cong J$ , where  $J$  is the simple Lie Algebra of derivations of  $\Gamma(Z/C)$ . In particular, if  $Z = C$ , then  $L$  is solvable. (Received January 30, 1956.)

557. Bjarni Jónsson. *Universal relational systems.*

Consider systems  $\mathfrak{A} = \langle A, R_0, \dots, R_n \rangle$  with  $R_i \subseteq A^{p(i)}$ ,  $n$  and  $p(i)$  being finite. The set-operations are applied termwise to such systems,  $\overline{\mathfrak{A}}$  denotes the cardinal of  $A$ , and  $\mathfrak{B} < \mathfrak{A}$  means that  $\mathfrak{B}$  is a subsystem of  $\mathfrak{A}$ , i.e., that  $\mathfrak{B} = \langle B, S_0, \dots, S_n \rangle$  where  $B \subseteq A$  and  $S_i = R_i \cap B^{p(i)}$ . Given a class  $K$  of systems, a system  $\mathfrak{A} \in K$  with  $\mathfrak{A} = \aleph_\alpha$  is said to be  $(\aleph_\alpha, K)$  universal if every  $\mathfrak{B} \in K$  with  $\overline{\mathfrak{B}} = \aleph_\alpha$  is isomorphic to a subsystem of  $\mathfrak{A}$ . Consider the following conditions on  $K$ : I. There exist  $\mathfrak{A}, \mathfrak{B} \in K$  which are not isomorphic. II. If  $\mathfrak{A} \cong \mathfrak{B} \in K$  then  $\mathfrak{A} \in K$ . III. For any  $\mathfrak{A}, \mathfrak{B} \in K$  there exists  $\mathfrak{C} \in K$  which contains subsystems isomorphic to  $\mathfrak{A}$  and  $\mathfrak{B}$ . IV. If  $\mathfrak{A}, \mathfrak{B}, \mathfrak{A} \cap \mathfrak{B} \in K$ ,  $\mathfrak{A} \cap \mathfrak{B} < \mathfrak{A}$  and  $\mathfrak{A} \cap \mathfrak{B} < \mathfrak{B}$ , then there exist  $\mathfrak{C} \in K$  with  $\mathfrak{A} < \mathfrak{C}$  and  $\mathfrak{B} < \mathfrak{C}$ . V. The union of any simply ordered subset of  $K$  is in  $K$ . VI $_\gamma$ . If  $\mathfrak{A} < \mathfrak{B} \in K$  and  $\overline{\mathfrak{A}} < \aleph_\gamma$ , then there exists  $\mathfrak{C} \in K$  with  $\mathfrak{A} < \mathfrak{C} < \mathfrak{B}$  and  $\overline{\mathfrak{C}} < \aleph_\gamma$ . *Results:* If I-V and VI $_0$  hold, then there exists an  $(\aleph_0, K)$  universal system. Under the assumption of the Generalized Continuum Hypothesis, if I-V and VI $_1$  hold, then there exists an  $(\aleph_\alpha, K)$  universal system for every  $\alpha > 0$ . I-V and VI $_1$  hold for the classes of all groups, groupoids, partially ordered systems, and lattices, but IV fails for semigroups and for distributive lattices. (Received March 5, 1956.)

558t. J. A. Kalman: *Absorption laws relating two binary associative operations.*

Let  $(i \dots j)$  be the family of all algebraic systems with binary associative operations  $\wedge$  and  $\vee$  related by the absorption laws  $(i), \dots, (j)$ , where (1)  $x \wedge (x \vee y) = x$ , (2)  $x \vee (y \wedge x) = x$ , (3)  $(y \vee x) \wedge x = x$ , (4)  $(x \wedge y) \vee x = x$ , (5)  $x \vee (x \wedge y) = x$ , (6)  $x \wedge (y \vee x) = x$ , (7)  $(y \wedge x) \vee x = x$ , (8)  $(x \vee y) \wedge x = x$ . Call two such families *similar* if the systems of one may be obtained from those of the other by interchanging the operations and/or reversing one or both pairs of operands. *Results.* Every such family is similar to exactly one of (1), (12), (13), (15), (16), (17), (18), (123), (125), (126), (127), (128), (135), (136), (1234) = (12356), (1235), (1236) = (1267), (1237), (1256), (1258), (1268), (1357), (1368), (12357). In each case the laws indicated, together with the associative laws, are independent postulates for the corresponding family. The family of lattices is (1234) = (12356) (cf. Bull. Amer. Math. Soc. Abstract 61-4-537, I). Jordan's postulates (Arch. Math. vol. 2 (1949) pp. 56-59) for the family (3456) are not independent, since (3456) = (345) = (356). (Research supported in part by the University of New Zealand Research Fund.) (Received March 5, 1956.)

559t. Marvin Marcus: *Singular values of a product of matrices.*

Let  $(a)$  be a  $k$ -tuple of numbers and  $E_r(a)$  be the  $r$ th elementary symmetric function of the  $(a)$ . Let  $A, B, AB$  be  $n$ -square complex matrices with singular values  $\alpha_i \geq \alpha_{i+1}$ ,  $\omega_i \geq \omega_{i+1}$ ,  $\lambda_i \geq \lambda_{i+1}$  respectively. *Results:* (i) For  $1 \leq r \leq k \leq n$ ,  $E_r(\lambda_1^2, \dots, \lambda_k^2) \leq \prod_{j=1}^r \alpha_j^2 E_r(\omega_1^2, \dots, \omega_k^2)$ ;  $E_r(\lambda_n^2, \dots, \lambda_{n-k+1}^2) \geq \prod_{j=1}^r \alpha_{n-j+1}^2 E_r(\omega_n^2, \dots, \omega_{n-k+1}^2)$ . (ii) Let  $X = AB$ ,  $\sigma \geq 0$ ,  $\delta \geq 0$ ,  $\sigma + \delta = 1$ . Let  $\lambda_i^2, \lambda_i^2 \geq \lambda_{i+1}^2$  be the eigenvalues of the convex sum  $\sigma X^*X + \delta XX^*$ . Then for  $1 \leq r \leq k \leq n$ ,  $E_r(\lambda_n^2, \dots, \lambda_{n-k+1}^2) \geq (\prod_{j=1}^r \alpha_{n-j+1}^2 \omega_{n-j+1}^2) \cdot E_r^2(\omega_n^2, \dots, \omega_{n-k+1}^2) E_r^\delta(\alpha_n^2, \dots, \alpha_{n-k+1}^2)$ . These results extend a recent theorem of A. Horn (*On the singular values of a product of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 374). (Received February 20, 1956.)

560. Marvin Marcus: *An extension of the results of K. Fan and H. Weyl concerning singular values.*

We use the notation of the above abstract (singular values of a product of ma-

trices). Theorem. Let  $A$  be an arbitrary complex  $n$ -square matrix with eigenvalues  $\lambda_i$ ,  $|\lambda_i| \geq |\lambda_{i+1}|$ . Let  $\sigma$ ,  $\delta \geq 0$ ,  $\sigma + \delta = 1$  and denote the eigenvalues of the convex sum  $\sigma A^*A + \delta AA^*$  by  $\alpha_i^2$ ,  $\alpha_i^2 \geq \alpha_{i+1}^2$ . If  $1 \leq r \leq k \leq n$  and  $s \geq 1$  then  $E_r(\alpha_1^{2s}, \dots, \alpha_k^{2s}) \geq E_r(|\lambda_1|^{2s}, \dots, |\lambda_k|^{2s})$ . H. Weyl obtained this result for  $\delta = 0$ ,  $k = n$ ,  $s > 0$ . (Inequalities between the two kinds of eigenvalues of a linear transformation, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 408). K. Fan obtained this result for  $r = k$ . (On a theorem of Weyl concerning the eigenvalues of linear transformations II, Proc. Nat. Acad. Sci. U.S.A. vol. 36 (1950) p. 31). (Received February 20, 1956.)

561. G. C. Preston: *Rings of continuous  $p$ -adic valued functions.* Preliminary Report.

Let  $Q_p$  represent the field of  $p$ -adic numbers,  $X$  a compact Hausdorff space, and  $C(X, Q_p)$  the ring of continuous functions from  $X$  to  $Q_p$ . If  $f \in C(X, Q_p)$ ,  $|f| = \max_{x \in X} |f(x)|$  is a norm on  $C(X, Q_p)$  which takes on the values  $p^n$  only. Under this definition,  $C(X, Q_p)$  is complete and is a topological  $Q_p$ -module (the mapping  $q \times f$  onto  $qf$  is a bilinear, continuous mapping of  $Q_p \times C(X, Q_p)$  onto  $C(X, Q_p)$ ). If there are sufficiently many continuous mappings of  $X$  into  $Q_p$ , ( $x \neq y$  implies there exists  $f$  such that  $f(x) \neq f(y)$ ) then all maximal ideals of  $C(X, Q_p)$  are "fixed" that is, a maximal ideal is of the form  $\{f: f(x) = 0\}$  for some  $x \in X$ . If  $X$  is completely regular with respect to  $Q_p$ ,  $X \cong \mathfrak{M}$  where  $\mathfrak{M}$  is the space of all maximal ideals of  $C(X, Q_p)$  with the standard topology. (See Gelfand and Silov, Mat. Sbornik, vol. 9 (1941) pp. 25-39). For the most part, proofs carry over from the theory of normed rings. Two questions which arise in this connection are the following: (1) Which  $Q_p$ -modules are representable as such function rings? (2) Which compact Hausdorff spaces are completely regular with respect to  $Q_p$ ? (Received February 27, 1956.)

562. Herman Rubin and R. L. San Soucie (p): *On the center of certain rings.*

Jacobson has proved (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228-245), using Schur's lemma and the concept of transformation center of a ring, that the center of a simple (not necessarily associative) ring  $R$  ( $R^2 \neq 0$ ) is either zero or a field. We offer a simple, short, direct proof of this result, using only elementary properties of a ring. Call a (not necessarily associative) ring  $R$  primitive in case  $R$  contains a maximal regular right ideal  $M$  containing no two sided ideals of  $R$  except the zero ideal. If  $R$  is associative, it is known that the center of  $R$  is a commutative integral domain. We give a very short proof of this fact in the general case, and believe the result to be new. Indeed, we prove more: a nonzero center element of  $R$  is not a zero divisor in  $R$ . Finally, we give a method for constructing an associative primitive ring whose center is an arbitrarily chosen commutative integral domain. (Received March 7, 1956.)

563. J. H. Walter: *A complete set of invariants for representation modules of algebras.*

Let  $\mathfrak{A}$  be an algebra with an identity, which is of finite dimension over a field  $K$ . We assume that  $K$  is algebraically closed. Let  $V$  be a representation module for  $\mathfrak{A}$  and let  $V = V_1 \supseteq V_2 \supseteq \dots \supseteq V_k = 0$  be its upper Loewy series. Associated with the factors  $V_i^* = V_i/V_{i+1}$  are the vector spaces  $\mathfrak{A}$ -Hom  $(F_i, V_i^*)$  of  $\mathfrak{A}$ -homomorphisms of  $F_i$  into  $V_i^*$ , where the  $F_i$  are the distinct irreducible representation modules of  $\mathfrak{A}$ . Let  $\mathfrak{A}$ -Hom $^*(F_i, V_i)$  denote the space dual to  $\mathfrak{A}$ -Hom  $(F_i, V_i)$ . Then it is shown that equiv-

alence classes of bilinear functions from  $\mathfrak{A}$ -Hom  $(F_i, V_r) \times \mathfrak{A}$ -Hom  $(F_j, V_s)$  to the first chain groups  $C^1(\mathfrak{A}, \text{Hom}(F_j, F_i))$ , introduced by Hochschild, provide a complete set of invariants for  $V$ . (Received April 25, 1956.)

564*t*. Morgan Ward: *Quadratic residues of Lucas functions.*

Let  $(U): 0, 1, P, \dots$  and  $(V): 2, P, P^2 - 2Q, \dots$  be the Lucas functions associated with the polynomial  $x^2 - Px + Q$ ,  $P$  and  $Q$  integers. Let  $p$  be a prime with positive rank of apparition  $r$  in  $(U)$ . The distribution of quadratic residues of  $(U)$  and  $(V)$  modulo  $p$  is studied, and in particular, conditions for the appearance of at least one nonresidue in  $(U)$  or  $(V)$  are found. The problem is trivial for  $r$  large (i.e.  $p/r < 9$ ). For  $r$  small (i.e.  $r < 9$ ) definitive results are obtained when  $P^2 - 4Q$  is a residue of  $p$ . The criteria for appearance of a nonresidue depend on the quadratic, cubic or biquadratic character modulo  $p$  of simple arithmetic functions of  $r$ . (Received March 15, 1956.)

ANALYSIS

565. E. J. Akutowicz: *The spectral resolution of Watson transforms.*

If  $\phi$  belongs to  $L^2(0, \infty)$ , the (complex) Watson transform  $W\phi$  of  $\phi$  is defined by  $\int_0^t W\phi(u) du = \int_0^\infty (k(tu)/u)\phi(u) du$ , where  $k(u)/u$  belongs to  $L^2(0, \infty)$ , and  $\int_0^\infty (k(t'u)k(t''u)/u^2) du = \min(t', t'')$ . The operator  $W$  is then unitary in  $L^2(0, \infty)$ . G. Doetsch (*Die Eigenwerte und Eigenfunktionen von Integraltransformationen*, Math. Ann. vol. 117 (1939)) has obtained several results pertaining to the point spectrum of  $W$ . In this note the full spectral resolution of  $W$  is obtained in explicit form. This is done by taking account of the behavior of the Poisson integral (with respect to the unit circle) arising from the operator  $Q_\lambda = W(W - \lambda I)^{-1}$ ,  $|\lambda| \neq 1$ . (Received March 5, 1956.)

566. F. H. Brownell: *Asymptotic distribution of eigenvalues for the lower part of the Schrodinger operator spectrum.*

From Weyl's law for the distribution of eigenvalues of the  $n$ -dimensional membrane problem it is to be expected that  $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$  satisfies  $N(\lambda) = [(2\pi^{1/2})^n \Gamma(2^{-1}n + 1)]^{-1} \cdot \int_{V(\mathbf{x}) \leq \lambda} [\lambda - V(\mathbf{x})]^{n/2} d\mu_n(\mathbf{x})$  asymptotically as  $N(\lambda) \rightarrow +\infty$  with increasing  $\lambda$  for the Schrodinger eigenvalue equation  $-\nabla^2 u_j(\mathbf{x}) + V(\mathbf{x})u_j(\mathbf{x}) = \lambda_j u_j(\mathbf{x})$  over  $\mathbf{x} \in R_n$ , euclidean  $n$ -space with  $n$ -dimensional Lebesgue measure  $\mu_n$ , with  $u_j \in L_2(R_n)$ . Recently this has been proved only for  $n=1$  by Titchmarsh subject to rather stringent smoothness conditions on  $V(\mathbf{x})$ , and in addition the condition that  $V(\mathbf{x})$  be monotone increasing in  $|\mathbf{x}|$  with limit  $+\infty$  as  $|\mathbf{x}| \rightarrow +\infty$ . We use Titchmarsh's results to obtain the stated asymptotic relation when  $n \geq 3$  for radial  $V(\mathbf{x}) = q(|\mathbf{x}|)$  under weaker conditions on  $q(r)$  over  $0 < r < +\infty$ . It should be possible to prove this relation for general  $V(\mathbf{x})$  without this radial restriction by using Courant's domain comparison methods and known perturbation results. (Received March 1, 1956.)

567*t*. E. A. Coddington: *On self-adjoint ordinary differential operators.*

Let  $L = p_0 D^n + p_1 D^{n-1} + \dots + p_n$ ,  $D = d/dx$ , where the  $p_k$  are complex-valued functions on an open real interval  $a < x < b$  of class  $C^{n-k}$ ,  $p_0(x) \neq 0$ . Assume that  $L$  coincides with its Lagrange adjoint. Let  $S$  be the operator in  $L^2(a, b)$  with domain  $C_0^\infty$ , the set of functions of class  $C^n$  vanishing outside some closed bounded subinterval of  $(a, b)$ , and put  $Su = Lu$ ,  $u \in C_0^\infty$ . Assuming that  $S$  has a self-adjoint extension  $H$ ,

it is shown how to define self-adjoint boundary value problems on closed bounded subintervals  $\delta$  of  $(a, b)$  in such a way as to produce, in the limit  $\delta \rightarrow (a, b)$ , the spectral matrix associated with the expansion theorem and Parseval equality for  $H$ . This matrix is related to Green's function for  $H-l$ ,  $l \neq 0$ , which is shown to be a limit of Green's functions for the problems defined on the subintervals of  $(a, b)$ . Finally it is shown how the spectral family of projections  $E(\lambda)$  associated with  $H$  can be represented in terms of the spectral matrix and solutions of  $Lu = \lambda u$ . This representation implies, among other things, the expansion theorem and Parseval equality. (Received February 27, 1956.)

568t. E. A. Coddington: *On maximal symmetric ordinary differential operators.*

The results of the preceding paper are extended to all maximal symmetric extensions  $H$  of  $S$ , which need not be self-adjoint. The generalized resolvent and generalized resolution of the identity (in the sense of M. A. Naimark) replace the resolvent and spectral family of projections, respectively. The domains of  $H$  and its adjoint  $H^*$  are characterized by certain boundary conditions. (Received February 27, 1956.)

569t. E. A. Coddington: *Some Banach algebras.*

Let  $\{\phi_k\}$  be a complete orthonormal set in an  $L^2$  space of complex-valued functions, and suppose, in addition, that  $\phi_k \in L^\infty \cap L^1$  for all  $k$ . If the complex constants  $\nu_k$  satisfy certain conditions, and multiplication is defined by  $f * g = \sum (f, \phi_k)(g, \phi_k)\nu_k\phi_k$  for  $f, g \in L^1$ , then it can be shown that  $L^1$  with this multiplication is a commutative Banach algebra  $A$  which is regular. These algebras contain simple examples of regular Banach algebras which are not self-adjoint. Under additional assumptions on the  $\phi_k$  the algebra  $A$  is semi-simple, and satisfies the hypothesis for the generalized Wiener theorem. (Received February 27, 1956.)

570t. I. I. Hirschman, Jr.: *A characterization of minimal projections which commute with left translations.* Preliminary Report.

Let  $G$  be a compact topological group with elements  $x, u$ , etc. A bounded linear transformation  $P$  of  $L^1(G)$  into itself such that  $P^2 = P$ ,  $P^* \subset P$  is called a projection. It is said to be minimal if  $\|P\| = 1$ . Let  $H$  be a closed subgroup of  $G$  and  $dm(u)$  Haar measure on  $H$  normalized by the condition  $m(H) = 1$ . Let  $g(u)$  be a rank 1 character on  $H$ . Then the formula  $Pf(x) = \int_H f(xu^{-1})g(u)dm(u)$  defines a minimal projection commuting with left translations. Conversely every minimal projection commuting with left translations is of this form. (Received March 5, 1956.)

571t. I. I. Hirschman, Jr.: *On the Young-Hausdorff-Riesz theorem for compact groups.* Preliminary Report.

Let  $G$  be a compact group and let, for each  $\alpha \in A$ ,  $[g_{i,j}^\alpha(x)]_{i,j=1,2,\dots,r(\alpha)}$  be an irreducible unitary representation of  $G$  of rank  $r(\alpha)$ ,  $A$  being a complete set of inequivalent representations. Let  $c_{i,j}^\alpha(f) = \int g_{i,j}^\alpha(x)f(x)dx$  where  $dx$  is Haar measure on  $G$  normalized by the condition that  $G$  has measure 1. The analogue of the Young-Hausdorff-Riesz theorem asserts that  $[\sum_{\alpha \in A} r(\alpha)^{2-q/2} (\sum_{i,j=1}^{r(\alpha)} |C_{i,j}^\alpha(f)|^2)^{q/2}]^{1/q} \leq \|f\|_p$  ( $1 \leq p \leq 2$ ,  $p^{-1} + q^{-1} = 1$ ). Let  $p$ ,  $1 < p < 2$ , be fixed. A function  $f(x)$  such that equality obtains in the above equality is said to be a maximal function in  $L^p(G)$ . Let  $H$  be an open and closed normal subgroup of  $G$ ,  $g(x)$  a rank one character on  $H$ ,  $u$  an

element of  $G$ , and  $c$  a complex constant. Then  $f(x) = c\chi_{uH}(x)g(u^{-1}x)$ , where  $\chi_{uH}(x)$  is the characteristic function of  $uH$ , is maximal in  $L^p(G)$ . Conversely every maximal function is of this form. The corresponding result for locally compact Abelian groups was obtained by E. Hewitt and the author in the Amer. J. Math. vol. 76 (1954) pp. 839-851. (Received March 5, 1956.)

572. W. Jurkat and A. Peyerimhoff (p): *On Fourier constants of functions in Lipschitz classes.*

A real-valued function  $f(t)$  with period  $2\pi$  is said to belong to  $\text{Lip}(k, p)$  ( $0 < k \leq 1, p \geq 1$ ), if  $f \in L^p$  and if  $\int_0^{2\pi} |f(t+h) - f(t)|^p dt = 0(h^p)$  for  $h \rightarrow 0$ . The notation  $f(t) \sim \sum_{-\infty}^{+\infty} c_n e^{int}$  will be used and it will be assumed that  $0 < k < 1, 1 < p \leq 2$ . The following theorem is proved: If  $f \in \text{Lip}(k, p)$ , then (\*)  $\sum_1^\infty n \alpha_n |n^k c_n|^q < \infty$  for all sequences  $0 < \alpha_n \searrow, \sum \alpha_n < \infty$  ( $1/p + 1/q = 1$ ), and all possible relations  $\sum_1^\infty n^\delta |\sum_{v \leq n} A_{n-v}^{\beta-1} v^\gamma c_v|^\gamma < \infty$  ( $\beta - 1, \gamma > 0$ ) are consequences of (\*). For  $p=2$  (\*) is equivalent with  $f \in \text{Lip}(k, 2)$ . This theorem contains results by Bernstein, H. C. Chow, Hardy-Littlewood, Sunouchi, Szász, Zygmund, concerning the convergence of  $\sum |c_n|^k$ ,  $\sum n^\delta |c_n|$  or the absolute Cesàro summability of  $\sum c_n$ . If  $f \in \text{Lip}(k, p)$ , and if  $c_n^*$  denotes the sequence  $|c_0|, |c_1|, |c_{-1}|, \dots$  arranged in decreasing order, then  $c_n^* = 0(n^{-k-1/q})$ . If, on the other hand,  $a_n \searrow$  and  $a_n = 0(n^{-k-1/q})$ , then  $\sum a_n \cos n\phi$  and  $\sum a_n \sin n\phi$  are Fourier series of functions of  $\text{Lip}(k, p)$ . This leads to the following theorem: A necessary and sufficient condition that a sequence  $\{c_n\}$  ( $\bar{c}_n = c_{-n}$ ) represents for some variation of the arguments and arrangement of the  $c_n$ 's a sequence of Fourier constants of a function  $f \in \text{Lip}(k, p)$  is  $c_n^* = 0(n^{-k-1/q})$ . There are similar theorems in the case  $p > 2$ . (Received March 7, 1956.)

573. V. A. Kramer: *Asymptotic perturbation series.*

Let  $H(t)$  be an operator defined on a Hilbert space by the series  $H(t) = H_0 + tH_1 + t^2H_2 + \dots$ . Suppose that  $\mathfrak{D}$ , the domain of  $H(t)$ , is dense, that  $H_0$  is self-adjoint and bounded below by 1, that  $H_i$  for  $i > 0$  is symmetric and bounded below by 0, and that  $t \geq 0$ . Then  $H(t)$  has a Friedrich's extension,  $\bar{H}(t)$ . Suppose that  $H_0$  is the Friedrich's extension of its own contraction to  $\mathfrak{D}$ . If  $\lambda_0$  is an isolated eigenvalue of  $H_0$  and is less than the smallest accumulation point of the spectrum of  $H_0$ , then  $\bar{H}(t)$  has an eigenvalue  $\lambda(t)$  which converges to  $\lambda_0$ . The method of identification of coefficients leads to formal series (possibly finite) for this eigenvalue and the associated eigenvectors;  $\lambda(t) = \lambda_0 + t\lambda_1 + t^2\lambda_2 + \dots, \phi(t) = \phi_0 + t\phi_1 + t^2\phi_2 + \dots$ . A set of conditions sufficient for these expansions to be asymptotically valid is derived by appeal to an earlier study of the series expansion of  $\bar{H}(t)^{-1}\phi$ . [See Bull. Amer. Math. Soc. Abstract 62-1-54.] The main result is that if  $\phi_i$  is defined for  $0 \leq i \leq M$  and  $\phi_i$  satisfies inverse conditions for order  $M-i$ , then the eigenvector expansion is valid to order  $M$ . An auxiliary theorem then allows us to expand the eigenvalue to order  $2M$ . Corresponding results are obtained for the degenerate case. Kato [J. Fac. Sci. Imp. Univ. Tokyo, Sect. I, vol. 5 (1951) pp. 145-225] has solved the problem for operators with only two terms using a different method. (Received March 7, 1956.)

574. L. B. Rall: *The extension of the quadratic formula to Banach spaces.*

Given a Banach space  $X$ , a symmetric bilinear operator  $B$  in  $X$ , a linear operator  $L$  in  $X$ , and a point  $v$  in  $X$ , the expression  $Qu = Buu + Lu + v = 0$  is called a quadratic equation in  $X$ . If  $2Bz + L = 0$ , or for  $(2Bz + L)^{-1}$  existing for some  $z$  in  $X$ , the quadratic

equation is transformed into  $Bxx+y=0$  or  $B'xx+Ix+y=0$ , respectively, where  $u=x+z$ . The set  $F(B)$  is composed of all elements  $w$  of  $X$  such that  $(Bw)^2-B(Bww)=0$ . The equation  $Bxx+y=0$  has a solution  $x$  in  $F(B)$  if and only if  $(-By)^{1/2}$  exists,  $Bx=(-By)^{1/2}$ , and  $(-By)^{1/2}x+y=0$ . The equation  $B'xx+Ix+y=0$  has a solution  $x$  in  $F(B')$  if and only if  $(I-4B'y)^{1/2}$  exists,  $B'x=-I/2+(I-4B'y)^{1/2}/2$ , and  $[I+(I-4B'y)^{1/2}]2x+y=0$ . (Research sponsored by the U. S. Army Office of Ordnance Research, under the terms of Contract DA 04-200-ORD-177.) (Received March 7, 1956.)

575. R. M. Redheffer: *On pairs of harmonic functions.*

A characteristic value  $k$  for a region  $R$  is a constant such that the problem  $\Delta u+ku=0$ ,  $u=0$  at the boundary, has a nontrivial solution. (a) Let  $u$  and  $v$  be harmonic functions which map a region  $R$  into a region  $R^*$  contained in a circle of radius  $r$ . Then  $\inf (u_x^2+u_y^2+v_x^2+v_y^2) \leq kr^2/2$  where  $k$  is the smallest characteristic value for  $R$ . (b) Let  $u$  and  $v$  be harmonic in a bounded region  $R$ , and let  $M$  and  $m$  be constant. Suppose  $u^2+v^2 \geq M$  at the boundary and  $u^2+v^2=m$  at an interior point,  $P$ . If  $d$  is the maximum distance from  $P$  to the boundary of  $R$ , then  $\sup (u_x^2+u_y^2+v_x^2+v_y^2) \geq 2(M-m)/d^2$ ,  $(x, y) \in R$ . That the latter result is sometimes optimum is shown by (c): Let  $u$  and  $v$  be twice-differentiable in a circle  $R$  of radius  $r$ . Suppose  $u^2+v^2 \geq M$  at the boundary and  $u^2+v^2=0$  at the center. Then the value of  $\inf_{(u,v)} \sup_{(x,y)} (u_x^2+u_y^2+v_x^2+v_y^2)$  is  $M/r^2$  when  $u$  and  $v$  are unrestricted; it is  $2M/r^2$  when  $u$  and  $v$  are harmonic; and it is  $2M/r^2$  when  $u$  and  $v$  are conjugate harmonic. (d) Let  $u(x, y)$  and  $v(x, y)$  be harmonic in a region  $R$ , and let their "conjugateness" be measured by  $\alpha+v_x^2=v_y^2-u_x^2-u_y^2$ ,  $\beta=2(u_xv_x+u_yv_y)$  (so that  $\alpha=\beta=0$  when  $u+iv$  is analytic). Let  $M$  and  $m$  be constants with  $M \geq m > 0$ . Suppose  $u^2+v^2 \geq M$  on the boundary,  $u^2+v^2=m$ , a minimum, at an interior point,  $P$ . Then  $\sup (\alpha^2+\beta^2)^{1/2} \geq (2m/D^2) \log (M/m)$  where  $D$  is the maximum diameter of  $R$ . The sharper inequality  $\sup [(\alpha^2+\beta^2)^{1/2}/(u^2+v^2)] \geq (2/d^2) \log M/m$  is also true, where  $d$  is the maximum distance from  $P$  to the boundary. (Received January 27, 1956.)

576t. R. M. Redheffer: *On entire solutions of nonlinear equations.*

An entire solution  $u(x, y)$  of a partial differential equation (\*) is a function which is twice-differentiable and satisfies \* at every point of the  $(x, y)$ -plane. (a) Let  $h(x)$  be continuous for  $-\infty < x < \infty$  and define  $H(t) = \int_0^t h(x) dx$ . Then the partial differential equation  $\Delta u = (u_x^2 + u_y^2)h(u)$  has a nonconstant entire solution  $u(x, y)$  if and only if the integrals  $\int_0^\infty e^{-H(t)} dt$ ,  $\int_0^\infty e^{-H(-v)} dt$  are both divergent. (b) Let  $r(t)$  be a continuous positive function for  $t \geq t_0$  and define  $R(t) = \int_{t_0}^t r(\xi) d\xi$ . If  $\int_{t_0}^\infty [R(t)]^{-1/2} dt < \infty$  then every entire solution of  $\Delta u \geq r(u)$  is constant. (c) Let  $u(x, y)$  be an entire solution of  $\Delta u \geq r(u) + s(u)(u_x^2 + u_y^2)$  such that  $\sup u = m < \infty$ . If there exists an  $\epsilon > 0$  such that  $r(u) + as(u) > 0$  for  $0 < a < \epsilon$ ,  $m - u < \epsilon$ , then  $u$  is constant. (d) Suppose there is a sequence of compact sets  $R_n$  such that  $(x^2 + y^2 \leq n^2) \subset R_n$  and such that  $\sup u / \log (x^2 + y^2) \rightarrow 0$  as  $n \rightarrow \infty$ , the sup being taken on the boundary of  $R_n$ . If  $\Delta u$  is an entire solution of  $\Delta u \geq 0$ , then  $u$  is constant. (Under the additional hypothesis that  $r(t)$  is twice-differentiable, increasing, and positively convex, the result (b) is due to Wittich. See Math. Zeit., 1944.) (Received January 27, 1956.)

577t. R. M. Redheffer: *On the growth of solutions of nonlinear equations.*

Write  $\rho^2 = x^2 + y^2$ ,  $\rho > 0$ , and define  $M(\rho) = \sup u(x, y)$  on  $x^2 + y^2 = \rho^2$ . Then (a)



For  $\rho \geq \rho_0$  and  $u > u_0$  let  $u$  be twice differentiable and satisfy  $\Delta u \geq r(u) + s(u)(u_x^2 + u_y^2)$ ,  $\limsup M(\rho)/\rho \leq 0$ ,  $\rho \rightarrow \infty$ . Suppose there is a fixed positive sequence  $a_i \rightarrow 0$  such that  $\liminf u_i [r(u_i)/a_i + a_i s(u_i)] > 0$  whenever  $u_i \rightarrow \infty$ . Then  $u$  is bounded from above for  $\rho \geq \rho_0$ . (b) For  $\rho \geq \rho_0$  and  $u > u_0$  let  $u$  be twice differentiable and satisfy  $\Delta u \geq r(u) + s(u)(u_x^2 + u_y^2)$ ,  $\limsup M(\rho)/e^\rho \leq 0$ ,  $\rho \rightarrow \infty$ . Suppose there is a constant  $A$  such that, for  $u > u_1$ , we have  $2(r(u)s(u))^{1/2} \geq 1 + 1/\log(Au)$  when  $u^2 s(u) \geq r(u)$ ,  $r(u)/u + us(u) \geq 1 + 1/\log(Au)$  when  $u^2 s(u) \leq r(u)$ ,  $0 \leq r(u)$ . Then  $u$  is bounded from above, for  $\rho \geq \rho_0$ . (c) Let  $v$  be a twice differentiable function satisfying  $\Delta v \leq 0$  exterior to a bounded region  $R$ , and let  $P$  be a point exterior to  $R$  and exterior to the circle  $\rho = 1$ . Then there is a simple polygonal arc, extending from  $P$  to infinity, upon which  $v/\log \rho$  is bounded from above. (d) For  $u > u_0$ ,  $v > v_0$  and  $\rho > \rho_0$  let  $u(x, y)$  and  $v(x, y) = v(\rho)$  be twice differentiable in an unbounded region  $R$  and satisfy  $v\Delta u \geq u\Delta v$ ,  $\sup u \leq A$  at the boundary,  $\lim v(\rho) = \infty$  as  $\rho \rightarrow \infty$ . If  $\limsup M(\rho)/v(\rho) = m > 0$  as  $\rho \rightarrow \infty$ , then  $\liminf M(\rho)/v(\rho) = m$ . Here  $M(\rho) = \sup u$ ,  $(x^2 + y^2 = \rho^2)$ ,  $(x, y) \in R$ . The hypothesis may be replaced by  $v\Delta u - u\Delta v \geq f(x, y, u, u_x, u_y, v_x, v_y)$  with mild restrictions on  $f$ . (Received January 27, 1956.)

578. W. M. Stone: *On a class of Bessel-like functions*. Preliminary Report.

Studies of the behavior of radar receiver systems have led to the 4th order differential equation  $(D^2 + m^2)(D^2 + m^{*2})y = y \exp(-2x)$ , where  $m$  and  $m^*$  are complex conjugates. Taking  $H_m(\lambda_n x) = y_m(\lambda_n x)y_{m^*}(\lambda_n) - y_{m^*}(\lambda_n x)y_m(\lambda_n)$ ,  $H'_m(\lambda_n) = 0$ , it follows that the set  $\{x^{1/2}H_m(\lambda_n x)\}$  is complete  $L^p(0, 1)$ ,  $1 \leq p < \infty$ ,  $\text{Re}(m) > -1/2$ . Many formulas and theorems, particularly the orthogonality property, are closely analogous to those of the Bessel functions. In particular, infinite sums and infinite products involving the  $\lambda_n$  are readily evaluated in closed form. (Received March 6, 1956.)

#### APPLIED MATHEMATICS

579. W. Karush ( $p$ ) and A. Vazsonyi: *A minimization problem involving a finite sequence and its first differences*.

The problem is to minimize  $F(x)$  where  $F(x) = \sum_{i=0}^n f_i(x_i) + \sum_{i=1}^n [\alpha \max\{x_i - x_{i-1}, 0\} - \beta \min\{x_i - x_{i-1}, 0\}]$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$ ,  $\alpha + \beta > 0$  are constants, and  $x = (x_0, x_1, \dots, x_n)$  is subject to  $a_i \leq x_i \leq b_i$ ,  $i = 0, 1, \dots, n$ . It is assumed that  $f_i(u)$  is continuous on the interval  $I_i: a_i \leq u \leq b_i$  and *ditonic* on  $I_i$ , i.e., for some  $m_i \in I_i$ ,  $f_i$  is nonincreasing on  $a_i \leq u \leq m_i$  and is nondecreasing on  $m_i \leq u \leq b_i$ . An algorithm is given for obtaining a minimizing vector  $x$ . The problem is first reduced to finding the minimum of a function  $G(t) = \sum_{i=0}^r g_i(t_i)$ ,  $g_i(u)$  continuous,  $\{t_i\}$  an arbitrary alternating sequence, i.e.,  $t_0 \leq t_1, t_1 \geq t_2$ , etc. (or  $t_0 \geq t_1, t_1 \leq t_2$ , etc.). The reduced index  $r$  equals the number of monotonic sections of  $\{m_i\}$ . The latter problem is then solved by iteration. For the cases when the  $f_i$  are (1) piecewise analytic, (2) piecewise linear, (3) convex, the method of solution variously simplifies. (Received March 12, 1956.)

580. W. R. Wasow: *On the accuracy of finite difference approximations to Dirichlet problems involving piecewise analytic boundary values*.

The known appraisals of the truncation error  $v = U - u$  involved in replacing the solution  $u$  of a boundary value problem for a partial differential equation by the solution  $U$  of a difference equation are either restricted to very special—usually rec-

tangular—domains, or they assume the boundedness of three or more partial derivatives of  $u$  in the whole domain  $R$  under consideration. This is unsatisfactory, since most computational problems deal with boundaries or boundary values that have corners. As a first step to overcome this difficulty it is proved that for the simplest finite difference approximation to Dirichlet's problem for Laplace's equation the order of the error is not affected by jumps in the first derivative of the boundary function. More precisely, it is assumed that the boundary  $C$  is a simple closed analytic curve, and that the boundary function  $f(s)$  is continuous and piecewise analytic. The approximating function  $U$  satisfies  $U(x+h, y) + U(x-h, y) + U(x, y+h) + U(x, y-h) = 4U(x, y)$ , if all five grid points are in  $R+C$ . At grid points near the boundary  $U$  is assigned the prescribed boundary value at a nearby point of  $C$ . Then the truncation error is  $O(h)$  except possibly near the boundary. The main tool is an asymptotic analysis of Green's function for the discrete problem. (Received January 3, 1956.)

### GEOMETRY

581. Iacopo Barsotti: *Abelian varieties over fields of positive characteristic.*

Let  $A$  be an  $n$ -dimensional nonsingular abelian variety over the algebraically closed field  $k$  of characteristic  $p \neq 0$ ; let  $\mathfrak{A}, \mathfrak{C}$  be the groups of the  $(n-1)$ -dimensional cycles on  $A$  which are, respectively, arithmetically equivalent ( $\equiv$ ) to 0, or algebraically equivalent ( $\approx$ ) to 0; let  $p^f$  be the number of the distinct  $P \in A$  such that  $P^p$  is the identity of  $A$ ; let  $\Gamma$  be the  $k$ -module of the factor sets of  $A$  into a 1-dimensional vector variety over  $k$ , and let  $\Gamma_0$  be the  $k$ -module of the elements of  $\Gamma$  which are associated to the identity of  $\Gamma$ . The following results are proved: (1)  $A$  has torsion 1, that is,  $\mathfrak{A} = \mathfrak{C}$ ; (2)  $\Gamma/\Gamma_0$  is a  $k$ -module of order  $n$ ; (3) the  $k$ -module of the differentials of the first kind on  $A$  has a  $k$ -independent basis of the form  $\{x_1^{-1}dx_1, \dots, x_f^{-1}dx_f, dx_{f+1}, \dots, dx_{f+e}, \omega_{f+e+1}, \dots, \omega_n\}$ , where  $x_i \in k(A)$ ,  $e > 0$  if  $f < n$ , and each  $\omega_i$  is not a linear combination, with coefficients in  $k$ , of differentials of the first kind of the types  $dx$  or  $x^{-1}dx$ . Results (1) and (2) are generalizations of similar results, previously found by the author, valid in the case of characteristic zero. By means of examples, it is proved that either relation,  $e+f < n$ , or  $e+f = n$ , may actually occur. (Received March 23, 1956.)

582. V. L. Klee, Jr.: *The structure of semispaces.*

When  $L$  is a real linear space and  $p$  a point of  $L$ , a "semispace" at  $p$  in  $L$  is a maximal convex subset of  $L \setminus \{p\}$ . This notion was recently introduced by Hammer, who showed that the class of all semispaces in  $L$  is the smallest intersection-base for the class of all convex proper subsets of  $L$ . The present note studies the structure of semispaces, and of sets which are the intersection of countably many semispaces. It is proved that each semispace in  $L$  is generated in a simple way by an ordered set of linear functionals, and  $L$  is of countable dimension if and only if each semispace can be generated by the set of "coördinate" functionals associated with a basis. For a convex subset  $C$  of a space of countable dimension, the following assertions are equivalent: (i)  $C$  is the intersection of a countable family of semispaces; (ii) whenever a family of convex sets has intersection  $C$ , so has some countable subfamily. (Received March 2, 1956.)

583. C. O. Oakley (p) and R. J. Wisner: *Flexagons*.

A flexagon (the paper model of which is a network of equilateral triangles folded into a flexible hexagon, discovered by R. Feynman, A. H. Stone, B. Tuckerman, and J. W. Tukey) is defined abstractly. Let  $m$  be a positive integer. For  $m=1$ , the single permutation of the integer 1 is called a pat of degree 1. For  $m=r+s>1$ , the permutation  $A_r A_{r-1} \cdots A_2 A_1 b_s b_{s-1} \cdots b_2 b_1$ , where  $A_i = a_i + s$ , of the integers 1 to  $m$  is called a pat of degree  $m$  if the permutation  $a_1 a_2 \cdots a_r$  of the integers 1 to  $r$  and the permutation  $b_1 b_2 \cdots b_s$  of the integers 1 to  $s$  are pats of degree  $r$  and  $s$ , respectively. A flexagon is an ordered pair of pats of degrees  $p$  and  $q$ , and the order  $N$  of the flexagon is  $N=p+q$  (this being the number of faces obtainable under a pinching operation). A basis of pats arises from which all flexagons are generated and models can be made from a single straight strip of equilateral triangles by folding and gluing. Equivalence of flexagons is defined and the number of equivalence classes of flexagons of order  $N$  is found to be  $V_{N-1}/N + \{V_K/2\} + \{2V_L/3\}$ , where  $V_M = \binom{2M-1}{M} / (2M-1)$  and  $N=2K=3L$  with inclusion of a braced term when and only when  $K$  or  $L$  is integrable. (Received March 6, 1956.)

## LOGIC AND FOUNDATIONS

584. C. C. Chang and Anne C. Davis (p): *Arithmetical classes closed under direct product.*

For terminology see Tarski, *Indag. Math.* vol. 16 (1954) pp. 572-578. As is known, (i) If a class  $K \subseteq AC_\delta$  and if  $K$  is characterized by conditional sentences, then  $K$  is closed under direct product (Horn, *J. Symbolic Logic* vol. 16 (1951) p. 17, Th. 4). Tarski asked whether the converse of (i) holds (cf. Bing, *Proc. Amer. Math. Soc.* vol. 6 (1955) p. 836). By I and II below, it is evident that the converse fails. Given a class of relational systems,  $\mathfrak{A}_i = \langle A_i, R_i \rangle$ , where  $i \in I$  and each  $R_i$  is a binary relation, a new system  $P'_i \in \mathfrak{A}_i = \langle A, R \rangle$  is defined: elements of  $A$  are equivalence classes of the functions  $f$  on  $I$  such that  $f(i) \in A_i$  for each  $i \in I$ ,  $f$  and  $g$  belonging to the same class whenever  $f(i) = g(i)$  for all but finitely many indices  $i$ ; two equivalence classes with representatives  $f$  and  $g$  are in the relation  $R$  iff  $\langle f(i), g(i) \rangle \in R_i$  for almost all  $i \in I$ . Clearly, the operation  $P'$  is definable on elements of an arbitrary similarity class. I. If  $K \subseteq AC_\delta$  and  $K$  is characterized by conditional sentences, then  $P'_i \in \mathfrak{A}_i \in K$  whenever  $I$  is infinite and  $\mathfrak{A}_i \in K$  for each  $i \in I$ . II. The class of atomistic Boolean algebras, although closed under direct product, is never closed under  $P'$ . (Received March 5, 1956.)

585t. William Craig: *Further remarks on finite axiomatizability using additional predicates.*

For notation see the two abstracts of Vaught's papers below. Theorem 1: If  $T$  is f.a.<sup>+</sup>, then  $T$  is strongly f.a.<sup>+</sup>. The proof uses Theorem (I) and Lemma (II) of Vaught's abstracts. Corollary: If  $S \in PC$  then also  $\phi(S) \in PC$  where  $\phi(S) = \bigcap \{X \mid S \subseteq X \in AC\}$ . Theorem 2: If  $T$  is axiomatizable and if some  $(ER_1) \cdots (ER_k)F$ ,  $F$  being a first-order sentence, has the same finite models as  $T$  and no infinite models other than those of  $T$ , then  $T$  is f.a.<sup>+</sup>. The proof depends on  $(K)$  only. Corollary: If  $T$  is not f.a.<sup>+</sup>, then the set  $T_f$  of those first-order sentences which are true of every finite model of  $T$  is not f.a.<sup>+</sup>. Theorem 3: If  $T$  has only a finite number of nonisomorphic finite models and if  $T$  is axiomatizable, then  $T$  is f.a.<sup>+</sup>. This is an easy extension of  $(K)$ . (Received April 23, 1956.)

586. Solomon Feferman: *Degrees of unsolvability correlated to theories with standard formalization.*

For terminology and notation see Tarski, Mostowski, Robinson, *Undecidable Theories*, and Post, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316. For any  $\alpha \subseteq \omega$ , the following conditions define a consistent theory  $T[\alpha]$  with standard formalization: Symbols of  $T[\alpha]$  are the unary predicate symbol  $A$  and the distinct individual constant symbols  $c_0, \dots, c_n, \dots$ . Axioms of  $T[\alpha]$  are the sentences: (i)  $(c_n \neq c_m)$  for  $n \neq m$ , (ii)  $\bigwedge x_0 \dots \bigwedge x_{n-1} \bigvee x_n \bigvee x_{n+1} [A(x_n) \wedge \sim A(x_{n+1}) \wedge \prod_{i < n} (x_i \neq x_n \wedge x_i \neq x_{n+1})]$  for any  $n$ , and (iii)  $A(c_n)$  for any  $n \in \alpha$ . Let  $\alpha^*$  be the set of Gödel numbers of those sentences  $\phi$  provable from the axioms of  $T[\alpha]$  in first order predicate calculus with identity. If  $\alpha$  is recursively enumerable, so is  $\alpha^*$ . The following theorems hold for arbitrary  $\alpha \subseteq \omega$ . (I)  $\alpha$  is 1-1 reducible to  $\alpha^*$  (II)  $\alpha^*$  is reducible to  $\alpha$  by unbounded truth tables. (III)  $\alpha$  and  $\alpha^*$  have the same degree of unsolvability. (IV)  $T[\omega]$  is consistent and decidable; hence  $T[\alpha]$  is not essentially undecidable. Theorem (II) is obtained by an application of the method of eliminating quantifiers (see, e.g., Tarski, *A Decision Method for Algebra and Geometry*, 2d ed. (1951) p. 50, Note 11). Using the announced solution of Post's problem by Friedberg (Bull. Amer. Math. Soc. Abstract 62-3-362) Theorem (III) provides a negative answer to a conjecture of Myhill, Zeitschr. f. math. Logik u. Grundlagen d. Math. vol. 1 (1955) p. 98. (Received February 28, 1956.)

587. L. A. Henkin: *A nondenumerable formal system.* Preliminary report.

Let  $F$  be a first-order functional calculus,  $\Gamma$  a set of its individual constants.  $F$  is  $\Gamma$ -complete if, for every formula  $B(x)$  containing  $x$  as sole free variable,  $\vdash B(\alpha)$  for all  $\alpha \in \Gamma$  implies  $\vdash (\forall x)B(x)$ .  $M$  is a  $\Gamma$ -model of  $F$  if all formal theorems of  $F$  are true of  $M$  and each element of  $M$  is denoted by a constant of  $\Gamma$ .  $F$  is  $\Gamma$ -saturated if every sentence of  $F$  which is true of all  $\Gamma$ -models is provable in  $F$ . We have previously shown that for denumerable systems  $F$ ,  $\Gamma$ -completeness and  $\Gamma$ -saturation are equivalent. We now have an example of a nondenumerable system  $F$  (and a set  $\Gamma$  having the same cardinality as the set of all formulas of  $F$ ) which is  $\Gamma$ -complete, has many  $\Gamma$ -models, but is not  $\Gamma$ -saturated. A formal notion (strong  $\Gamma$ -completeness) is developed, which is equivalent to the semantical concept of  $\Gamma$ -saturation for all consistent systems. (Received March 5, 1956.)

588. R. L. Vaught: *Finite axiomatizability using additional predicates.* I.

Consider theories  $T$ , formalized in classical first order logic with identity, having finitely many predicates.  $T$  is called *finitely axiomatizable using additional predicates* (f.a.+ ) if  $T$  has an extension  $T'$  such that (i)  $T'$  has finitely many additional predicates, (ii)  $T'$  is finitely axiomatizable, and (iii) a sentence of  $T$  is valid in  $T'$  iff it is valid in  $T$ . If, moreover, all models of  $T$  are obtainable from models of  $T'$  by dropping the additional relations,  $T$  is called *strongly f.a.+*. (Thus in Tarski's terminology (Indag. Math. vol. 16 (1954) p. 584),  $T$  is strongly f.a.+ iff the  $AC_\delta$  of models of  $T$  is a PC.) Kleene's work (Memoirs of the Amer. Math. Soc., no. 10, 1952) established: (K) *If all models of  $T$  are infinite, then  $T$  is axiomatizable (=recursively axiomatizable) iff  $T$  is f.a.+*. By the device of using "finite set theory with individuals" where Kleene used "number theory"—a possibility noted by Tarski—we obtain the stronger: Theorem (I). *If all models of  $T$  are infinite, then  $T$  is axiomatizable iff  $T$  is strongly f.a.+*. A corollary is an

affirmative answer to question (iii) of Łoś (Coll. Math. vol. 3 (1954) p. 62) for the case of axiomatizable theories, having only infinite models. (Received March 5, 1956.)

589t. R. L. Vaught: *Finite axiomatizability using additional predicates. II.*

For notation see preceding abstract. Lemma (II). *If  $T'$  is any extension of  $T$  satisfying (i), (ii), and (iii), then all finite models of  $T$  are obtainable from models of  $T'$  by dropping the additional relations.* Theorem (III). *If  $T$  is f.a.<sup>+</sup>, and if  $T^*$ , the extension of  $T$  obtained by adding axioms excluding all finite models, is decidable (in particular, if  $T$  is  $\aleph_\alpha$ -categorical) then  $T$  is decidable.* (III) slightly generalizes a result of Henkin (Indag. Math. vol. 17 (1955) p. 326). From a recent result of Tarski (J. Symbolic Logic vol. 19 (1954) p. 158) follows: Theorem (IV). *In (I) and in Kleene's result on logic without identity, one additional binary predicate suffices.* Using (II) and an idea of Ehrenfeucht, one sees that the problem of characterizing those theories which are f.a.<sup>+</sup>, when restricted to theories having only the identity predicate, virtually coincides with an (unsolved) problem proposed by Scholz (J. Symbolic Logic vol. 17 (1952) p. 160). Some partial results (unpublished) of Mostowski on the Scholz problem may, therefore, be translated, yielding, in particular, the fact that *the hypothesis of (K) cannot be omitted.* (Received March 5, 1956.)

#### STATISTICS AND PROBABILITY

590. E. G. Kimme: *On the convergence of sequences of stochastic processes.*

Let  $\{x_n(t), 0 \leq t \leq 1\}$ ,  $n=1, 2, \dots$  be a sequence of separable stochastic processes with independent increments such that  $p\{x_n(0)=0\}=1$ . Let  $\{x(t), 0 \leq t \leq 1\}$  be a separable stochastic process with independent increments, no fixed points of discontinuity, and such that  $p\{x(t)=0\}=1$ . Two modes of convergence of  $\{x_n(t), 0 \leq t \leq 1\}$  to  $\{x(t), 0 \leq t \leq 1\}$  are defined. The stronger of these, called uniform convergence in distribution, suffices for the convergence in distribution of  $F[x_n(\cdot)]$  to  $F[x(\cdot)]$ , for certain classes of functionals, among which is  $F[f]=\sup_{0 \leq t \leq 1} f(t)$ . It is shown that the usual ordinary convergence in distribution of the sequence of stochastic processes under discussion implies uniform convergence in distribution thereof when the processes all have stationary increments as well. These results are used to obtain an extension of Gnedenko's necessary and sufficient condition for the weak convergence of sums of independent random variables to the case where  $x_n(t) = \sum_{j=1}^{k_n t} x_{nj}$ ,  $\{x_{nj}, 1 \leq j \leq k_n\}$ ,  $n=1, 2, \dots$ , being a sequence of finite sequences of infinitesimal independent random variables ("independent within rows"). This generalizes some results of Donsker, Kac and Erdős, Udagawa, and others. (Received March 22, 1956.)

591. R. B. Leipnik: *Statistics of irreversible processes.*

A stochastic model is set up for irreversible processes of transport and collision of particles in the presence of radiation and external parameters. Time-independent distributions are found under a variety of conditions, generalizing results of Moyal. Time-dependent distributions are found under more restrictive conditions, and questions of ergodicity, convergence to equilibrium, and recurrence times are answered. This is related to the work of Kac on the Ehrenfest model. Applications are made to the theory of imperfect gases and liquids, following ideas of H. S. Green. (Received March 5, 1956.)

592t. A. T. Bharucha-Reid: *On the fundamental theorem concerning branching processes.*

Let  $F(s) = \sum_{x=0}^{\infty} p_x s^x$ ,  $|s| < 1$ , be the generating function of the probabilities  $p_x$  associated with the discrete branching process  $\{X_n, n \geq 0\}$ . The fundamental theorem can be stated as follows: Let  $p_0 > 0$ , and let  $\omega = \Pr [X_n = 0]$  for some  $n$ . Then  $\omega = 1$  if  $\sum x p_x \leq 1$  while, if  $\sum x p_x > 1$ ,  $\omega$ , is given by the smallest non-negative root of the functional equation  $F(s) = s$ . Various authors have given proofs based on Markov chain theory, fixed point methods, etc. In this note a theorem due to Kantorovitch (Acta Math. vol. 71 (1939) pp. 63-97) is used to establish the existence of a solution  $\omega \in [0, 1]$ , and to show that  $\omega$  can be obtained by the method of successive approximations. (Received March 7, 1956.)

#### TOPOLOGY

593. E. A. Michael: *Some new characterizations of paracompactness.*

Call a collection  $\mathcal{G}$  of subsets of a topological space *closure-preserving* if, for every subcollection  $\mathcal{B} \subset \mathcal{G}$ , the union of the closures is the closure of the union. (Any locally finite collection is closure-preserving, but not conversely.) It is shown that, both in the usual definition of a paracompact space, and in the characterizations obtained by the author in [Proc. Amer. Math. Soc. vol. 4 (1953) pp. 831-838], "locally finite" can be replaced by "closure-preserving." This implies that the image of a paracompact space, under a closed, continuous mapping, must be paracompact, thus settling the author's Research Problem [Bull. Amer. Math. Soc. Research Problem 61-6-29]. (Received February 16, 1956.)

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